Doctoral Thesis

Automorphisms of quartic surfaces and Cremona transformations

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Abstract

Any Cremona transformation of the three-dimensional projective space $\mathbb{P}^3$ that stabilizes a smooth quartic surface $S \subset \mathbb{P}^3$ induces an automorphism of S. The converse problem, determining which automorphisms of S are restrictions of Cremona transformations of the ambient $\mathbb{P}^3$, remains an open question.

In this thesis, we provide a complete solution to this problem for quartic surfaces with Picard rank two. Building upon the theory of K3 surfaces and the Sarkisov program, we carry out a detailed analysis of the geometry of smooth quartic surfaces with Picard rank two, focusing on the restrictions imposed on the Picard lattice by the possible automorphisms and by Sarkisov links initiated from the blowup of curves on the quartics.

Keywords: Quartic K3 surfaces, Automorphisms, Cremona transformations, Calabi-Yau pairs, Sarkisov program, Lattices.

Resumo

Qualquer transformação de Cremona de $\mathbb{P}^3$ que estabilize uma superfície quártica suave $S \subset \mathbb{P}^3$ induz um automorfismo em S. O problema reverso, de determinar quais automorfismos de S são restrições de transformações de Cremona do espaço ambiente $\mathbb{P}^3$, continua sendo uma questão em aberto.

Nesta tese, fornecemos uma solução completa para este problema para superfícies quárticas com posto de Picard igual a dois. Com base na teoria das superfícies K3 e no programa de Sarkisov, realizamos uma análise detalhada da geometria das superfícies quárticas suaves com posto de Picard dois, concentrando-nos nas restrições impostas ao reticulado de Picard pelos possíveis automorfismos e pelos links de Sarkisov iniciados a partir do blowup de curvas sobre as quárticas.

Palavras-chave: Superfícies K3 quárticas, Automorfismos, Transformações de Cremona, Pares de Calabi-Yau, Programa de Sarkisov, Reticulados.

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Contents

Chapter 1

Introduction

In this thesis, we delve into the theory of K3 surfaces and the Sarkisov program to tackle the problem: "which automorphisms of a smooth quartic surface $S \subset \mathbb{P}^3$ are restrictions of Cremona transformations of $\mathbb{P}^3$" (see Problem 1 for more details). These areas form a cornerstone in the broader purpose of algebraic geometry: classifying projective varieties.

Birational geometry, a key area in algebraic geometry, investigates when two varieties are birationally equivalent, i.e. they are isomorphic outside lower-dimensional closed subsets. The *Minimal Model Program* (MMP) aims to identify the "simplest" representatives for each birational class of projective varieties, often termed *minimal models*. This program simplifies the birational classification of projective varieties by focusing the study on the resulting outputs. In the one-dimensional case, every complex algebraic curve is known to be birationally equivalent to a unique smooth projective curve, which are classified by their genus. The Italian school resolved the case of algebraic surfaces, proving that every algebraic surface is birationally equivalent to a smooth projective surface. Minimal models of smooth projective surfaces can be constructed using Castelnuovo's contraction theorem, while the Enriques classification organizes minimal surfaces into eight distinct classes based on their Kodaira dimension and numerical birational invariants. Among these, K3 surfaces belong to the four classes with Kodaira dimension zero.

A smooth complex projective surface S is a *K3 surface* if it has irregularity $h^1(S, \mathcal{O}_S) = 0$ and trivial canonical divisor $K_S \sim \mathcal{O}_S$. Consequently, S admits a unique (up to scalar) nowhere-vanishing holomorphic 2-form ω_S. Notable examples of K3 surfaces include smooth quartic surfaces in $\mathbb{P}^3$ and smooth double covers of $\mathbb{P}^2$ branched along smooth sextics.

The second cohomology group $H^2(S, \mathbb{Z})$ of a K3 surface S possesses both a lattice and a Hodge structure, with the latter determined by its *period line* $H^{2,0}(S) = \mathbb{C}\omega_S$. According to Kodaira, all complex K3 surfaces are diffeomorphic and share an isometric cohomology group $H^2(S, \mathbb{Z})$. Thus, K3 surfaces are distinguished by their complex structures or, equivalently, their period lines. The Weak Torelli Theorem asserts that isometric Hodge structures of two K3 surfaces imply isomorphic surfaces. Additionally, if the ample cones are also identified by the same isometry φ, the Global Torelli Theorem asserts that the isomorphism between them is uniquely determined by φ.

This interplay between K3 surfaces, lattices, and Hodge structures is profoundly influential. Geometric properties and automorphisms of K3 surfaces can be studied using lattice theory. The automorphism group $\mathrm{Aut}(S)$ of a K3 surface S is discrete and often infinite. Automorphisms are categorized as symplectic or non-symplectic based on their action on the 2-form ω_S. Symplectic automorphisms act trivially on ω_S, while non-symplectic ones do not. These concepts, introduced by Nikulin [Nik79], are pivotal in understanding how automorphisms influence the lattice structure of $H^2(S,\mathbb{Z})$. Building on this framework, numerous researchers have classified finite-order automorphisms and explored K3 surfaces admitting such structures. Key contributions to this field include [Nik83, Muk88, OZ98, MO98, AS08, OZ11, Tak11, AST11, GS13].

The Picard group $\mathrm{Pic}(S)$ of a K3 surface S also has a lattice structure and can be viewed as a sublattice of $H^2(S,\mathbb{Z})$. Isometries of $H^2(S,\mathbb{Z})$ that preserve its Hodge structure correspond to isometries of $\mathrm{Pic}(S)$ that extend appropriately to $H^2(S,\mathbb{Z})$ under a *Gluing condition*. By the Global Torelli Theorem and results of Nikulin [Nik83], the finite index subgroup $\mathrm{Aut}^{\pm}(S)$ of $\mathrm{Aut}(S)$ consisting of symplectic and *anti-symplectic* automorphisms (the latter are automorphisms acting as $-\,\mathrm{id}$ on ω_S) can be identified with isometries of $\mathrm{Pic}(S)$ that preserves the ample cone and extend to $H^2(S,\mathbb{Z})$ acting as $\pm\,\mathrm{id}$ on ω_S.

The complexity of the classification problem increases in higher dimensions and its development requires a modern version of the MMP. In dimension three, it was completed by Mori [Mor88], and more recently, Birkar, Cascini, Hacon, and McKernan [BCHM10] achieved a major breakthrough in higher dimensions. Key new ideas and techniques introduced in higher dimensions include the allowance of certain singularities and small modifications of varieties, such as *flips, flops and antiflips*. One of the two classes of outcomes of the MMP consists of *Mori fiber spaces* (MFS), which emerge as the program's end product for uniruled varieties. However, the outputs of the MMP depend on specific choices, which makes it important to study birational maps between these outputs within the same birational class. For MFS, the *Sarkisov program* offers an algorithmic approach to decompose birational maps into simpler ones, called *Sarkisov links*. The Sarkisov program was developed by Corti [Cor95] in dimension three, and latter extended to higher dimensions by Hacon and McKernan [HM13]. Later on, a version of the Sarkisov program for Calabi-Yau pairs was established by Corti and Kaloghiros [CK16]: the *volume preserving Sarkisov program*. It allows us to factorize any *volume preserving* birational map between *Mori fibered Calabi-Yau pairs* into *volume preserving Sarkisov links*.

The Sarkisov program has significantly contributed to understanding the birational self-maps of MFS, including the *Cremona group* $\mathrm{Bir}(\mathbb{P}^n)$, which is the group of birational self-maps of $\mathbb{P}^n$. This approach facilitates the study of the structure and the construction of special subgroups of $\mathrm{Bir}(\mathbb{P}^n)$. Recent advancements in this area can be found in works such as [LZ20, BLZ21, BSY22, Zik23a] and [ACM23].

1.1 On a problem of Gizatullin

It is natural to ask whether the automorphisms of a projective variety $X \subset \mathbb{P}^{n+1}$ can be expressed as restrictions of automorphisms of the ambient space $\mathbb{P}^{n+1}$. In this case, the automorphisms of the variety X can be expressed in coordinates, helping their description and the study of the geometry of X. Matsumura and Monsky [MM64] and Chang [Cha78], demonstrated that for smooth hypersurfaces $X \subset \mathbb{P}^{n+1}$ of degree d, every automorphism of X is induced by an automorphism of $\mathbb{P}^{n+1}$, except in the cases where $(n, d) = (1, 3)$ or $(2, 4)$.

In the exceptional case $(n, d) = (1, 3)$, corresponding to smooth elliptic curves $C \subset \mathbb{P}^2$, the automorphisms of C that are restrictions of automorphisms of $\mathbb{P}^2$ form a finite subgroup within the infinite group $\mathrm{Aut}(C) = C \rtimes \mathbb{Z}_m$, where $m \in \{2, 4, 6\}$. However, every automorphism of C can still be expressed as the restriction of a birational self-map of $\mathbb{P}^2$ [Ogu12, Theorem 2.2]. The other exceptional case, $(n, d) = (2, 4)$, involves smooth quartic surfaces $S \subset \mathbb{P}^3$. Here, the automorphism group $\mathrm{Aut}(S)$ is discrete and often infinite, as S is a K3 surface. Nonetheless, the subgroup of automorphisms induced by regular maps of $\mathbb{P}^3$ is finite. This naturally leads to the question of whether a similar result applies to quartic surfaces. This question lies within the context of the following problem posed by Gizatullin.

Problem 1 (Gizatullin)**.** Let $S \subset \mathbb{P}^3$ be a smooth quartic surface and $f \in \mathrm{Aut}(S)$ be an non-trivial automorphism of S. Does there exist a Cremona transformation φ of $\mathbb{P}^3$ which stabilizes S, i.e. $\varphi(S) = S$, and such that $\varphi|_S = f$?

When $S \subset \mathbb{P}^3$ is a smooth quartic surface with Picard rank $\rho(S) = 1$, this problem is trivially solved since $\mathrm{Aut}(S) = \{1\}$ (see Proposition 5.0.1). When the surface has higher Picard rank, Gizatullin's problem was first addressed by Oguiso who constructed the following two examples [Ogu12, Ogu13]. The first one is a smooth quartic surface $S \subset \mathbb{P}^3$ with $\rho(S) = 2$, $\mathrm{Aut}(S) \cong \mathbb{Z}$ and no non-trivial automorphism is induced by a Cremona transformation of $\mathbb{P}^3$ (see Example 5.0.2). The second one is a smooth quartic surface $S \subset \mathbb{P}^3$ with $\rho(S) = 3$, $\mathrm{Aut}(S) \cong \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ and every automorphism is realized as birational maps of the ambient space $\mathbb{P}^3$ (see Example 5.0.3). Based on these examples, a natural question posed by Oguiso in [Ogu12] is:

Problem 2 (Oguiso)**.** Is every automorphism of finite order of any smooth quartic surface $S \subset \mathbb{P}^3$ induced by a Cremona transformation of $\mathbb{P}^3$?

This thesis builds upon Oguiso's approach by employing results from the theory of K3 surfaces and the Sarkisov program to address Problem 1. Specifically, the solution to Gizatullin's problem is divided into two main theorems below, which were derived in collaboration with Carolina Araujo, Ana Quedo, and Sokratis Zikas [PQ25, APZ24]. The statements of these results have been slightly refined compared to their original form in the corresponding papers.

In his first example, Oguiso relied on a result from Takahashi (see Proposition 4.1.4), which is a consequence of the Sarkisov program. This result asserts that the existence of a non-regular Cremona transformation of $\mathbb{P}^3$ stabilizing a smooth quartic surface S forces the existence of a curve $C \subset S$ of degree < 16 that is not a complete intersection. In [PQ25], we exploit this idea and prove that when the quartic surface $S \subset \mathbb{P}^3$ has Picard rank $\rho(S) = 2$, the existence of such curves can be inferred from the lattice $\mathrm{Pic}(S)$. More precisely, we observe that the structure of the lattice $\mathrm{Pic}(S)$ is determined by the *discriminant* $r(S)$ of S, which is defined as the opposite of the determinant of a matrix associated to the intersection product on S. We then prove that when $r(S) > 233$, S does not contain such curves. As a result, the automorphisms of the quartic surface could, in principle, only be induced by regular maps of $\mathbb{P}^3$.

On the other hand, Matsumura and Monsky proved that any automorphism of S induced by an automorphism of $\mathbb{P}^3$ must have finite order (see [MM64, Theorem 1], [Ogu13, Theorem 3.2]). Moreover, automorphisms of K3 surfaces with finite order have been extensively studied through their action on the second cohomology group, as previously mentioned. In particular, we apply the classification by Nikulin [Nik83] for involutions and by Artebani, Sarti, and Taki [AST11] for automorphisms of higher order, to prove that any non-trivial finite order automorphism of a smooth quartic surface $S \subset \mathbb{P}^3$ with Picard rank two is an involution uniquely associated with a double cover $S \longrightarrow \mathbb{P}^2$. Therefore, when $r(S) > 233$, no non-trivial automorphisms can be induced by regular maps of $\mathbb{P}^3$.

This first analysis suggests that a classification for Gizatullin's problem is achievable in the case of Picard rank two.

Theorem A (Theorem 5.1.8)**.** Let $S \subset \mathbb{P}^3$ be a smooth quartic surface with Picard rank $\rho(S) = 2$ and discriminant $r(S) > 233$. Then the only automorphism of S arising from a Cremona transformation of $\mathbb{P}^3$ is the identity.

This result generalizes the first example of Oguiso, since the surfaces in that example have discriminant greater than 233 (see Example 5.0.2). It also provides the first negative answer to Problem 2 (see 5.1.1). The full analysis of Gizatullin's problem for Picard rank 2 is completed in [APZ24], where we present our second main theorem. This result leads to both positive and negative answers to Problem 1. For the notion of Aut-general, refer to Definition 3.4.9.

Theorem B (Proposition 5.2.3 + Theorem 5.2.13)**.** Let $S \subset \mathbb{P}^3$ be a smooth quartic surface with Picard rank $\rho(S) = 2$.

(1) If $r(S) > 57$ or $r(S) = 52$, then the only automorphism of S arising from a Cremona transformation of $\mathbb{P}^3$ is the identity.

(2) If $r(S) \leq 57$, $r(S) \neq 52$ and S is Aut-general, then every automorphism of S is induced by a Cremona transformation of $\mathbb{P}^3$. Moreover, one of the following holds.
   - $r(S) \in \{9, 12, 16, 24, 25, 33, 36, 44, 49, 57\}$ and $\mathrm{Aut}(S) = \{1\}$;
   - $r(S) \in \{17, 41\}$ and $\mathrm{Aut}(S) \cong \mathbb{Z}_2$;
   - $r(S) \in \{28, 56\}$ and $\mathrm{Aut}(S) \cong \mathbb{Z}_2 * \mathbb{Z}_2$; or
   - $r(S) \in \{20, 32, 40, 48\}$ and $\mathrm{Aut}(S) \cong \mathbb{Z}$.

This second part focuses on the Sarkisov decomposition of non-regular Cremona transformations of $\mathbb{P}^3$ stabilizing quartics. The volume-preserving version is particularly relevant, as the pair $(\mathbb{P}^3, S)$, where $S \subset \mathbb{P}^3$ is a smooth quartic surface, is a Mori fiber Calabi-Yau pair and every Cremona transformation stabilizing S is volume preserving. This version imposes strong conditions on the first Sarkisov link in any volume-preserving Sarkisov decomposition of a non-regular Cremona transformation stabilizing S. Specifically, these first Sarkisov links begin with the blowup of $\mathbb{P}^3$ along a curve $C \subset S$.

Sarkisov links starting from $\mathbb{P}^3$ are not fully classified. Some results in this direction can be found in [CM13] and [BL12], which classify curves in $\mathbb{P}^3$ such that their blowup X is a weak Fano variety and gives rise to Sarkisov links. They provide a list of pairs (g, d) of genus g and degree d of such curves. An interesting property is that all of these curves are contained in quartic surfaces [BL12, Proposition 2.8]. A first result in the case where X is not weak Fano appears in [Zik23b], where smooth curves in $\mathbb{P}^3$ lying in a smooth cubic surface whose blowup generates Sarkisov links are classified.

For the purpose of Gizatullin's problem, we analyze the blowup of curves C in $\mathbb{P}^3$, which are contained in a smooth quartic surface S with $\rho(S) = 2$. We prove that any curve C for which its blowup $X \to \mathbb{P}^3$ initiates a Sarkisov link satisfies that X is weak Fano, and the pair (g, d) appears in the list of Blanc and Lamy. The existence of such curves $C \subset S$ with certain (g, d) yields restrictions on $\mathrm{Pic}(S)$ and hence on the discriminant $r(S)$, allowing us to conclude the first part of Theorem B. Furthermore, the value of (g, d) alone is not sufficient for C to initiate a Sarkisov link and ensure that X is weak Fano. According to Blanc and Lamy, additional conditions on C and on the anticanonical morphism on X are required. We reformulate these conditions to make them easily verifiable from the Picard lattice $\mathrm{Pic}(S)$.

By studying the isometries of $\mathrm{Pic}(S)$ through the classical theory of binary quadratic forms, Galluzzi, Lombardo and Peters [GLP10] proved that when S is a K3 surface with Picard rank two, the finite index subgroup $\mathrm{Aut}^{\pm}(S) \subset \mathrm{Aut}(S)$ has four possible structures: $\{1\}$, $\mathbb{Z}_2$, $\mathbb{Z}$ or $\mathbb{Z}_2 * \mathbb{Z}_2$. Each case is determined by the existence of certain numerical classes in $\mathrm{Pic}(S)$. Furthermore, Lee [Lee23] described the action of $\mathrm{Aut}^{\pm}(S)$ on $\mathrm{Pic}(S)$, which allows us to determine its generators. Under the assumption in part two of Theorem B, where S is Aut-general, $\mathrm{Aut}^{\pm}(S) = \mathrm{Aut}(S)$, allowing us to compute the automorphism group and find its generators for each discriminant.

After describing the action of the generators of $\mathrm{Aut}(S)$ on $\mathrm{Pic}(S)$ when $r(S) \leq 57$ and $r(S) \neq 52$, we proceed to construct Cremona transformations realizing them. This is done by exploring the Sarkisov links initiated by blowing up certain curves $C \subset S$.

The thesis is structured as follows: In Chapter 2, we present some preliminaries, including notions from Lattice Theory, standard results on Intersection Theory, and various cones of divisors. We also provide an overview of the Minimal Model Program. In Chapter 3, we discuss fundamental results about K3 surfaces and their automorphisms, particularly focusing on the case of K3 surfaces with Picard rank two. In Chapter 4, we introduce the geometry of log Calabi-Yau pairs and the volume-preserving version of the Sarkisov Program, exploring the Sarkisov links initiated by the blowup of curves lying in smooth quartics, especially for the case of Picard rank two. In Chapter 5, we apply the theory of K3 surfaces and their automorphisms to the specific case of quartic surfaces with Picard rank two, proving Theorem A and Theorem B, and providing a counterexample to Problem 2. Finally, in Chapter 6, we discuss open questions arising from this work.

Chapter 2

Preliminaries

This chapter provides a concise introduction to the theory of lattices, algebraic geometry, intersection theory, and the Minimal Model Program. We present fundamental definitions and key results, providing proofs or references as needed, to lay the groundwork for subsequent discussions.

2.1 Background on lattices

Lattices are central to the theory of K3 surfaces, playing a crucial role in encoding both their algebraic and geometric properties. By linking topology, algebra, and complex geometry, lattices provide a foundational framework for understanding automorphisms, moduli spaces, divisors, and special structures on K3 surfaces. In this section, we introduce the theory of lattices, emphasizing their fundamental definitions and key results. We also discuss sublattices and isometries. For a more detailed and rigorous introduction to lattices, we refer to [Huy16, Chapter 14].

A *lattice* L is a free $\mathbb{Z}$-module of finite rank, equipped with a symmetric bilinear form $b_L : L \times L \longrightarrow \mathbb{Z}$. For $K = \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$, we denote the K-vector space $L_K := L \otimes_{\mathbb{Z}} K$. The bilinear form b_L extends naturally to L_K and is denoted by the same symbol. The *discriminant* of L is defined as $\operatorname{disc}(L) := \det(Q)$, where Q is the matrix representation of b_L with respect to any basis of L. If $\operatorname{disc}(L) \neq 0$, the lattice L is called *non-degenerate*, and if $\operatorname{disc}(L) = \pm 1$, it is called *unimodular*. The lattice L is said to be *even* if $b_L(x, x) \in 2\mathbb{Z}$ for any $x \in L$. The *signature* of a non-degenerate lattice L is the pair (l_+, l_-), where l_+ (resp., l_-) denotes the multiplicity of the eigenvalue 1 (resp., -1) for the quadratic form on $L_{\mathbb{R}}$.

From now on, let L be a non-degenerate even lattice. A *sublattice* of L is a $\mathbb{Z}$-submodule $L' \subset L$ such that the restriction of b_L to $L' \times L'$ is non-degenerate. A sublattice L' is called *primitive* if the quotient L/L' is torsion-free. The *dual lattice* of L is defined as

$$L^\vee = \operatorname{Hom}_{\mathbb{Z}}(L, \mathbb{Z}) \cong \left\{ x \in L \otimes_{\mathbb{Z}} \mathbb{Q} \,\middle|\, \forall y \in L\,,\; b_L(x, y) \in \mathbb{Z} \right\}.$$

There is a natural embedding of L in $L^\vee$ given by $x \mapsto b_L(x, \cdot)$. The *discriminant group* of L is the quotient

$A(L) = L^\vee/L$. This is a finite group, and the minimal number of generators is denoted by $l(A(L))$. The $\mathbb{Q}$-extension of b_L to $L^\vee$ is a symmetric bilinear form $L^\vee \times L^\vee \longrightarrow \mathbb{Q}/\mathbb{Z}$, which in turn induces a symmetric bilinear form b_A on $A(L)$:

$$b_A(x + L, y + L) = b_L(x, y) \mod \mathbb{Z}, \text{ for all } x, y \in L^\vee.$$

The *quadratic form* of L, $q_L \colon A(L) \longrightarrow \mathbb{Q}/2\mathbb{Z}$, is defined as

$$q_L(x + L) = b_L(x, x) \mod 2\mathbb{Z}, \text{ for all } x \in L^\vee.$$

Proposition 2.1.1. Let L be a non-degenerate lattice. Then

1. The index of L in $L^\vee$ is $|\operatorname{disc}(L)|$, i.e., $A(L)$ is a finite abelian group of order $|\operatorname{disc}(L)|$.
2. $l(A(L)) \leq \operatorname{rank}(L)$.
3. If L' is a sublattice of L with same rank, then $|\operatorname{disc}(L')| = [L : L']^2 \cdot |\operatorname{disc}(L)|$, and L' has the same signature as L.

Proof. We refer to [BHPVdV04, Lemma I.2.1] for the proof of (3). In order to prove (1) and (2), let $\{e_i\}$ be a basis of L and Q_L be a matrix representing the bilinear form with respect to this basis. Define $\{e_i'\}$, the dual basis for $L^\vee$, i.e., $e_i'(e_i) = \delta_{ij}$, where $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ otherwise. One can verify that the columns of Q^{-1} correspond exactly to the dual basis $\{e_i'\}$. Consequently, the matrix representing the bilinear form on $L^\vee$ with respect to this dual basis is $Q_{L^\vee} = Q^{-1}QQ^{-1} = Q^{-1}$. Therefore, $\operatorname{disc}(L^\vee) = 1/\operatorname{disc}(L)$. The result in (1) follows by applying (3) to $L \subset L^\vee$. Similarly, (2) follows from the observation that $\operatorname{rank}(L) = \operatorname{rank}(L')$ and the classes of the generators e_i' generate $A(L)$. □

The following are a few classical examples of lattices.

Example 2.1.2. Any lattice L of rank one can be expressed as $\langle m \rangle$, where $m \in \mathbb{Z} \setminus \{0\}$. In this case, L is isomorphic to $\mathbb{Z}$ and the bilinear form is given by $b(x, y) = mxy$. This lattice has discriminant $\operatorname{disc}(\langle m \rangle) = m$, discriminant group $A(\langle m \rangle) = \mathbb{Z}_m$, and signature (1) if $m > 0$, or (-1) if $m < 0$.

Example 2.1.3 (Hyperbolic plane)**.** We denote by U the lattice of rank two, generated by elements $e, f \in U$ such that the bilinear form is given by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This is an even unimodular lattice of signature $(1, 1)$, discriminant $\operatorname{disc}(U) = -1$ and trivial discriminant group $A(U)$.

Example 2.1.4 (Rescaling)**.** Let L be an even, non-degenerate lattice of signature (l_+, l_-), and let m be a non-zero integer. We define the rescaling lattice $L(m)$ as follows. As $\mathbb{Z}$-module, it is the same L, but the bilinear form $b_{L(m)}$ is given by $b_{L(m)}(x, y) = mb_L(x, y)$, for all $x, y \in L$. The lattice $L(m)$ has same rank as L, discriminant $\operatorname{disc}(L(m)) = m^{\operatorname{rank}(L)} \operatorname{disc}(L)$, and signature (l_+, l_-) or (l_-, l_+) depending on whether m is positive or not, respectively.

Some special cases are the following. The rank one lattice $\langle m\rangle$ is a rescaling of the lattice $\langle 1\rangle$, specifically: $\langle m\rangle = \langle 1\rangle(m)$. Moreover, the hyperbolic plane U, when rescaled by m, forms the lattice $U(m)$. It has discriminant m^2 and discriminant group $\mathbb{Z}_m \times \mathbb{Z}_m$.

Example 2.1.5 (E_8 lattice)**.** It is the lattice of rank 8, with intersection product given by the following matrix

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

The lattice E_8 is an even unimodular, with discriminant $\operatorname{disc}(E_8) = 1$ and signature $(8, 0)$.

Example 2.1.6 (Direct sum of lattices)**.** Let L and M be lattices with bilinear forms b_L and b_M, and signatures (l_+, l_-) and $(m_+ + m_-)$, respectively. We define the lattice $L \oplus M$ as follows. As a group, it is the direct sum $L \times M$ of L and M, and its bilinear form is given by

$$b_{L\oplus M}((x_1, y_1), (x_2, y_2)) = b_L(x_1, x_2) + b_M(y_1, y_2).$$

If Q_L and Q_M are matrices representing b_L and b_M, respectively. Then, $b_{L\oplus M}$ is representing by the matrix

$$Q_{L\oplus M} = \begin{pmatrix} Q_L & 0 \\ 0 & Q_M \end{pmatrix}.$$

Therefore, lattice $L \oplus M$ has rank $\operatorname{rank}(L) + \operatorname{rank}(M)$, signature $(l_+ + m_+, l_- + m_-)$ and discriminant $\operatorname{disc}(L \oplus M) = \operatorname{disc}(L) \cdot \operatorname{disc}(M)$. Moreover, there are natural embedding of L and M into $L \oplus M$ defined by $l \mapsto (l, 0)$ and $m \mapsto (0, m)$, respectively. Theses embeddings preserve the bilinear forms on L, M and $L \oplus M$.

Example 2.1.7 (K3 lattice)**.** The $K3$-lattice is defined as

$$\Lambda_{K3} := U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} = U \oplus U \oplus U \oplus E_8(-1) \oplus E_8(-1).$$

By Examples 2.1.6 and 2.1.4, this is an even unimodular lattice of signature $(3, 19)$ and discriminant -1.

Let L be an even unimodular lattice. By Proposition 2.1.1, the discriminant group $A(L)$ of L is trivial, implying $L = L^\vee$. Let $L_1 \subset L$ be a primitive sublattice and define the *orthogonal complement* of L_1 in L as the primitive sublattice $L_1^\perp = \{x \in L | b_L(x, y) = 0 \ \ \forall y \in L_1\}$.

Proposition 2.1.8. Let L be a unimodular lattice, L_1 be a primitive sublattice and $L_2 = L_1^\perp$. Then

1. $L_1 \oplus L_2$ is a sublattice of L with the same rank as L.
2. There is a natural group identification

$$L/(L_1 \oplus L_2) \cong A(L_1) \cong A(L_2). \tag{2.1}$$

3. The index of $L_1 \oplus L_2$ in L is given by $[L : L_1 \oplus L_2] = |\operatorname{disc}(L_1)| = |\operatorname{disc}(L_2)|$.

Proof. Clearly, $L_1 \oplus L_2$ is a sublattice of L. To see that $\operatorname{rank}(L) = \operatorname{rank}(L_1) + \operatorname{rank}(L_2)$, it suffices to consider the $\mathbb{Q}$-vector spaces $(L_1 \oplus L_2)_{\mathbb{Q}} = (L_1)_{\mathbb{Q}} \oplus (L_2)_{\mathbb{Q}}$ and $L_{\mathbb{Q}}$ and observe that they have the same dimension. This follows from the fact that the bilinear form b_L induces a non-degenerate inner product on $L_{\mathbb{Q}}$. This proves (1).

On the other hand, using the identification $\operatorname{Hom}_{\mathbb{Z}}(L, \mathbb{Z}) \cong L^\vee$ for any lattice L, we construct the following homomorphism

$$L \hookrightarrow L^\vee \cong \operatorname{Hom}_{\mathbb{Z}}(L, \mathbb{Z}) \to \operatorname{Hom}_{\mathbb{Z}}(L_1, \mathbb{Z}) \cong L_1^\vee \to A(L_1),$$

where the map $L^\vee \to L_1^\vee$ is surjective since $L_1 \subset L$ is primitive, and the kernel is $L_1 \oplus L_2$. Similarly, we have a map $L \to A(L_2)$ with same kernel. This provides the desired isomorphisms in (2.1). Finally, assertion (3) follows from (1) and (2). □

Let L and M be lattices of the same rank n. An *isometry* of L to M is an isomorphism $\varphi\colon L \to M$ preserving the bilinear forms. More precisely, $b_M(\varphi(x), \varphi(y)) = b_L(x, y)$ for every $x, y \in L$, and similarly with the inverse map φ^{-1}. If $\{e_i\}$ and $\{e'_i\}$ are bases of the lattices L and M, denote by Q_L and Q_M the matrices that represent the respective bilinear forms with respect to these bases. Any isometry $\varphi\colon L \to M$ can also be represented by a matrix, which we denote by the same symbol, by abuse of notation: $\varphi = (a_{ij})_{1 \le i,j \le n}$ where the entries are given by the relation $\varphi(e_j) = \sum a_{ij} e'_i$. Furthermore, the matrix satisfies $Q_L = \varphi^T \cdot Q_M \cdot \varphi$.

If $M = L$, we simply say that φ is an isometry of L. The *orthogonal group* of L, denoted by $O(L)$, is the group of all isometries of L. Any isometry $\varphi \in O(L)$ can be extended by linearity to an isometry of the dual lattice $L^\vee$. This extended isometry respects the pairing in $L^\vee$ and, therefore, descends naturally to an automorphism $\overline{\varphi}$ of the discriminant group $A(L) = L^\vee/L$.

The following is a necessary and sufficient condition to determine whether an isometry of a lattice acts trivially (up to sign) on the discriminant group.

Lemma 2.1.9. Let L be an even, non-degenerate lattice and $\varphi \in O(L)$ be an isometry. Denote by Q_L a matrix representing the bilinear form b_L, associated to a basis. Then, $\overline{\varphi} = \operatorname{id}$ (resp. $\overline{\varphi} = -\operatorname{id}$) on $A(L)$ if and only if $(\varphi - \operatorname{id}) * Q_L^{-1}$ (resp. $(\varphi + \operatorname{id}) * Q_L^{-1}$) is an integer matrix.

Proof. Recall that the columns of the matrix Q_L^{-1} form a basis of the dual lattice $L^\vee$. Furthermore, since $A(L)$ is defined as the quotient $L^\vee/L$, any isometry φ of L acts as id on $A(L)$ if and only if the image of each such generator under $\varphi - \operatorname{id}$ belongs to L. In other words, this occurs if and only if if the images of the generators under $\varphi - \operatorname{id}$ have integer coefficients when expressed in the basis of L. Similarly, φ acts as $-\operatorname{id}$ on $A(L)$ if and only if the image of each generator under $\varphi + \operatorname{id}$ belongs to L. □

When φ is an isometry of a unimodular lattice L preserving a sublattice L_1, the restriction $\varphi_{L_1} = \varphi|_{L_1}$ is an isometry of L_1. Moreover, φ preserves the orthogonal complement L_2 of L_1, and so, the restriction $\varphi_{L_2} = \varphi|_{L_2}$ is an isometry of L_2. In fact, these isometries satisfy the following nice property.

Corollary 2.1.10. Let L be an even unimodular lattice, L_1 be a primitive sublattice and $L_2 = L_1^{\perp}$. Suppose that $\varphi \in O(L)$ preserves both L_1 and L_2, so the restriction $\varphi_{L_i} = \varphi|_{L_i}$ to L_i is an element of $O(L_i)$ for $i = 1, 2$. Then, $\overline{\varphi_{L_1}} = \overline{\varphi_{L_2}}$ under the identification $A(L_1) \cong A(L_2)$ of (2.1).

Proof. Denote by $\alpha \colon A(L_1) \xrightarrow{\sim} A(L_2)$ the isomorphism in (2.1). The conjugation by α defines a bijection between $\mathrm{Aut}(A(L_1))$ and $\mathrm{Aut}(A(L_2))$. Consequently, the statement follows directly. □

The following proposition gives the converse. A proof of this can be found in [Nik80, Theorem 1.6.1, Corollary 1.5.2].

Proposition 2.1.11 (Gluing isometries)**.** Let L, L_1 and L_2 be as in Corollary 2.1.10, and let $\varphi_{L_1}, \varphi_{L_2}$ be isometries of L_1 and L_2, respectively. If $\overline{\varphi_{L_1}} = \overline{\varphi_{L_2}}$ under the identification $A(L_1) \cong A(L_2)$ of (2.1), then there exists an isometry φ on L whose restrictions to L_1 and L_2 are φ_{L_1} and φ_{L_2}, respectively.

Denote by $O(L, L_1) \subset O(L)$ the group of isometries of L preserving L_1. As a consequence of Corollary 2.1.10 and Proposition 2.1.11, we have the following identification:

$$O(L, L_1) \cong \{(\varphi_{L_1}, \varphi_{L_2}) \in O(L_1) \times O(L_2) | \overline{\varphi_{L_1}} = \overline{\varphi_{L_2}} \text{ under the identification } A(L_1) \cong A(L_2)\}. \tag{2.2}$$

Remark 2.1.12. An isometry φ_{L_1} of a primitive sublattice $L_1 \subset L$ whose action on $A(L_1)$ is $\overline{\varphi_{L_1}} = \pm\,\mathrm{id}$ can be naturally extended to L. Indeed, If $\overline{\varphi_{L_1}} = \mathrm{id}$ (resp. $\overline{\varphi_{L_1}} = -\,\mathrm{id}$), we define an isometry of $L_2 = L_1^{\perp}$ as $\varphi_{L_2} = \mathrm{id}$ (resp. $\varphi_{L_2} = -\,\mathrm{id}$). Since their actions coincide on the discriminant groups, the gluing of φ_{L_1} and φ_{L_2} gives an isometry of L.

Definition 2.1.13 (Invariant and co-invariant lattices)**.** Let φ be an isometry of a unimodular lattice L. The *invariant lattice* L^{φ} and the *co-invariant lattice* L_{φ} are defined as the lattices:

$$L^{\varphi} := \{x \in L | \varphi(x) = x\} \quad \text{and} \quad L_{\varphi} := (L^{\varphi})^{\perp}.$$

Lemma 2.1.14. Let φ be an isometry of a lattice L of finite order n. The following properties hold:

1. Both L^{φ} and L_{φ} are primitive sublattices of L.
2. L^{φ} contains the set $\{x + \varphi(x) + \cdots + \varphi^{n-1}(x) | x \in L\}$.
3. L_{φ} contains the set $\{x - \varphi^{i}(x) | x \in L,\ 0 \leq i < n\}$.
4. $L/(L^{\varphi} \oplus L_{\varphi})$ is of n-torsion.

Proof. Let $x \in L$ such $kx \in L^{\varphi}$ for some integer $k \geq 1$. Since $kx = \varphi(kx) = k\varphi(x)$, we deduce that $k(x - \varphi x) = 0$. Thus, $x - \varphi(x) = 0$, which implies $x \in L^{\varphi}$. Therefore, L^{φ} is a primitive sublattice of L, and consequently, L_{φ} is also a primitive sublattice. This is (1). The proof of (2) follows directly from the fact that φ has order n.

Now, let $x \in L$ and $y \in L^{\varphi}$. Observe that $b_L(x, y) = b_L(\varphi^i(x), \varphi^i(y)) = b_L(\varphi^i(x), y)$ for any $0 \leq i < n$. Thus, $b_L(x - \varphi^i(x), y) = 0$, and so, we get (3). Finally, to prove (4), let $x \in L$. Then,

$$nx = \sum_{i=0}^{n-1} \varphi^i(x) + \sum_{i=0}^{n-1} (x - \varphi^i(x)) \in L^{\varphi} \oplus L_{\varphi}.$$

□

We point out that if φ is the trivial isometry, i.e., $\varphi(x) = x$ for all $x \in L$, then $L^{\varphi} = L$ and $L_{\varphi} = \varnothing$.

2.1.1 Lattices of rank two

In this subsection, we focus on lattices of rank two, with special attention given to hyperbolic lattices of rank two. Hyperbolic lattices arise in the study of K3 surfaces, particularly in relation to the Picard lattice of a K3 surface, where such lattices may correspond to the Picard group of the surface.

Definition 2.1.15 (Hyperbolic lattice). A *hyperbolic lattice* is an even, non-degenerate lattice L of $\operatorname{rank}(L) > 1$ and signature $(1, \operatorname{rank}(L) - 1)$. By abuse of language, a lattice L of rank one with signature (1) is also called hyperbolic.

Classic examples of hyperbolic lattices of rank two are the lattices U and $U(m)$ with $m > 0$.

For the remainder of this section, we will establish the following notation.

Let L be a hyperbolic lattice of rank two and $\{e_1, e_2\}$ be a basis. In this basis, the bilinear form can be represented by a matrix

$$Q_L = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}, \tag{2.3}$$

where $a, b, c \in \mathbb{Z}$. Since L has signature $(1, 1)$, the discriminant of L is negative, i.e., $\operatorname{disc}(L) = 4ac - b^2 < 0$. For any two elements $x, y \in L$, we use the conventional notation $x \cdot y := b_L(x, y)$ and $x^2 := b_L(x, x)$. The following lemma provides a criterion, based on $\operatorname{disc}(L)$, for determining whether the lattice contains elements with a specific self-intersection.

Lemma 2.1.16. Let $L = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ be a hyperbolic lattice with bilinear form given by the matrix Q_L in (2.3). Let $r := -\operatorname{disc}(L) > 0$ and k be an integer number. Then, the following assertions hold:

1. For any $x = me_1 + ne_2$ in L we have that $2ax^2 = (x \cdot e_1)^2 - rn^2$.
2. There exist elements $x \in L$ such that $x^2 = 0$ if and only if r is a square.
3. If the Generalized Pell equation $u^2 - rv^2 = 4ak$ has no integer solutions, then there exist no an element $x \in L$ such that $x^2 = 2k$.

Proof. Let $x = me_1 + ne_2$ in L, for some $n, m \in \mathbb{Z}$, with $x^2 = 2k$. So, we can write

$$4ak = 2ax^2 = 4an^2 + 4abnm + 4acm^2 = 4an^2 + 4abnm + 4acm^2 + b^2m^2 - b^2m^2 = (x \cdot e_1)^2 - rm^2.$$

This gives us (1). Moreover, we see that $x^2 = 0$ if and only if r is a square number. Finally, we obtain assertion (3) by contraposition. □

An element $x \in L$ is called *primitive* if the rank one lattice $\langle x \rangle$ is a primitive sublattice of L. Equivalently, x is primitive if there is no $z \in L$ such that $x = kz$, for some integer $k > 1$. The following lemma guarantees that we can extend $\{x\}$ to a basis of L.

Lemma 2.1.17. Let L be a hyperbolic lattice of rank two, and let $x \in L$ be such that $x^2 = 2a$. If there is no integer $k > 1$ such that k^2 divides a, then x is a primitive element, and we can write $L = \mathbb{Z}x \oplus \mathbb{Z}w$ for some element w. In particular, there exist some $b, c \in \mathbb{Z}$ such that the intersection product is given by the following matrix

$$\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}. \tag{2.4}$$

Proof. We observe that x is a primitive element of the lattice; otherwise, there would exist an element $z \in L$ and an integer $k > 1$ such that $x = kz$. Thus, $k^2 z^2 = 2a$, which implies that k^2 divides a since z^2 is even.

Therefore, we can write $x = \alpha e_1 + \beta e_2$, for any basis $\{e_1, e_2\}$ of L, where $\gcd(\alpha, \beta) = 1$. Thus, there are integers γ and δ such that $\delta\alpha + \gamma\beta = 1$. Let $y := -\gamma e_1 + \delta e_2$. The matrix

$$A = \begin{pmatrix} \alpha & \beta \\ -\gamma & \delta \end{pmatrix}$$

is invertible over $\mathbb{Z}$ with determinant 1. Therefore, A is an isometry of L sending the basis $\{e_1, e_2\}$ to $\{x, y\}$. After this change of basis, the intersection matrix is the desired one. □

Now we stablish a condition on whether the discriminant of L determines the lattice.

Proposition 2.1.18. Let a be an integer satisfying the condition

$$n^2 \equiv m^2 \mod 4a \quad \text{implies} \quad n - m \equiv 0 \mod 2a \;\text{ or }\; n + m \equiv 0 \mod 2a. \tag{2.5}$$

Then, for any two hyperbolic lattices L and L' of rank two such that their intersection matrices are given by

$$Q_L = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} \qquad \text{and} \qquad Q_{L'} = \begin{pmatrix} 2a & b' \\ b' & 2c' \end{pmatrix}, \tag{2.6}$$

L is isometric to L' if and only if $\operatorname{disc}(L) = \operatorname{disc}(L')$.

Proof. One direction follows since any two isometric lattices have same discriminant. For the other direction, notice that from (2.5), $\frac{b-b'}{2a} \in \mathbb{Z}$ or $\frac{b+b'}{2a} \in \mathbb{Z}$. Define the map $\phi\colon L \to L'$ by the matrix

$$\phi = \begin{pmatrix} 1 & \frac{b \mp b'}{2a} \\ 0 & \pm 1 \end{pmatrix}.$$

That is, if $\{e_1, e_2\}$ and $\{e_1', e_2'\}$ are bases of L and L', respectively, such that the bilinear forms are represented by Q_L and $Q_{L'}$, then $e_1 \mapsto e_1'$ and $e_2 \mapsto \frac{b \mp b'}{2a} e_1' \pm e_2'$. This is an isometry from L to L'. Indeed, from

$4ac - b^2 = \operatorname{disc}(L) = \operatorname{disc}(L') = 4ab' - (b')^2$ we can check that

$$\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{b \mp b'}{2a} & \pm 1 \end{pmatrix} \begin{pmatrix} 2a & b' \\ b' & 2c' \end{pmatrix} \begin{pmatrix} 1 & \frac{b \mp b'}{2a} \\ 0 & \pm 1 \end{pmatrix}.$$

□

Now, we investigate isometries of lattices of rank two.

Let L be a hyperbolic lattice of rank two, and let $\{e_1, e_2\}$ be a basis such that the bilinear form is represented by the matrix (2.3). We can always assume that $c \neq 0$. Indeed, if $a = 0 = c$, then $b \neq 0$, $L \cong U(b)$ and $\{e_1, e_1 + e_2\}$ is a basis of L with $(e_1 + e_2)^2 = 2b \neq 0$. Thus we do a basis change if necessary. If $c = 0$ and $a \neq 0$, we write Q_L in the basis $\{e_2, e_1\}$. Now, take an isometry $\varphi \in O(L)$. With respect to the basis $\{e_1, e_2\}$, the associated matrix to φ has the following form

$$\varphi = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

Since φ is an isometry, we have that $\varphi^T Q_L \varphi = Q_L$ and so we obtain the equations

$$2a\alpha^2 + 2b\alpha\gamma + 2c\gamma^2 = 2a, \tag{2.7}$$

$$2a\alpha\beta + b\beta\gamma + b\alpha\delta + 2c\gamma\delta = b, \tag{2.8}$$

$$2a\beta^2 + 2b\beta\delta + 2c\delta^2 = 2c. \tag{2.9}$$

Lemma 2.1.19. Let $\varphi \in O(L)$ be a non-trivial isometry of order two. Then, with respect to the basis $\{e_1, e_2\}$, either $\varphi = -\operatorname{id}$, $L^\varphi = \{0\}$ and $L_\varphi = L$, or φ has the form

$$\varphi = \begin{pmatrix} \alpha & \beta \\ \frac{a}{c}\beta - \frac{b}{c}\alpha & -\alpha \end{pmatrix},$$

where (α, β) is an integer solution of the quadratic equation

$$\alpha^2 - \frac{b}{c}\alpha\beta + \frac{a}{c}\beta^2 = 1. \tag{2.10}$$

In this case, $\det(\varphi) = -1$, and both L^φ and L_φ has rank one.

Proof. From $\varphi^2 = \operatorname{id}$ we get the following equations

$$\alpha^2 + \beta\gamma = 1 = \delta^2 + \beta\gamma \quad \text{and} \quad \beta(\alpha + \delta) = 0 = \gamma(\alpha + \delta). \tag{2.11}$$

If $\alpha + \delta \neq 0$, we get $\beta = 0 = \gamma$ which implies that $\alpha = \delta = \pm 1$. Therefore, $\varphi = \pm \operatorname{id}$. Clearly the invariant lattice $L^\varphi = L$ when $\varphi = \operatorname{id}$. If $\varphi = -\operatorname{id}$, by Lemma 2.1.14, L_φ contains elements of the form $x - \varphi(x)$, for every $x \in L$. Thus, $2x \in L_\varphi$ and so $\operatorname{rank}(L_\varphi) = \operatorname{rank}(L)$. Then $L_\varphi = L$ since L_φ is a primitive sublattice of L.

If $\alpha + \delta = 0$, $\delta = -\alpha$. In this case, $\gamma = \frac{a}{c}\beta - \frac{b}{c}\alpha$ from (2.9) and (2.11). Moreover, the eigenvalues of φ are $\sqrt{\alpha^2 + \beta\gamma} = 1$ and $-\sqrt{\alpha^2 + \beta\gamma} = -1$ and $\det(\varphi) = -1$. □

Lemma 2.1.20. Let $\varphi \in O(L)$ be a non-trivial isometry of order four. Then $\det(\varphi) = 1$, $L^{\varphi} = \{0\}$ and $L_{\varphi} = L$.

Proof. By assumption, the isometry φ satisfies that $\varphi^4 = \text{id}$ and $\varphi^2 \neq \text{id}$. Thus, we get the following equations

$$(\alpha^2 + \beta\gamma)^2 + \beta\gamma(\alpha + \delta)^2 = 1 = (\delta^2 + \beta\gamma)^2 + \beta\gamma(\alpha + \delta)^2 \tag{2.12}$$

and

$$\beta(\alpha + \delta)(\alpha^2 + 2\beta\gamma + \delta^2) = 0 = \gamma(\alpha + \delta)(\alpha^2 + 2\beta\gamma + \delta^2). \tag{2.13}$$

- If $\alpha + \delta \neq 0$,
  - If $\alpha^2 + 2\beta\gamma + \delta^2 \neq 0$, from (2.13), $\beta = 0 = \gamma$ and then $\varphi = \pm\,\text{id}$. This is not possible.
  - If $\alpha^2 + 2\beta\gamma + \delta^2 = 0$, since $\det(\phi) = \alpha\delta - \beta\gamma = \pm 1$, we have that $\alpha^2 + 2\alpha\delta + \delta^2 = \pm 2$, which implies that $(\alpha + \delta)^2 = \pm 2$. This is not possible.
- If $\alpha + \delta = 0$, $\delta = -\alpha$. From (2.12), $(\alpha^2 + \beta\gamma)^2 = 1$ and since $\varphi^2 \neq \text{id}$, $\alpha^2 + \beta\gamma = -1$. This implies that $\det(\varphi) = 1$. Moreover, to determine L^{φ} we look at the eigenvalues of φ, which are i and $-i$. Therefore we have the desired result.

□

2.1.2 p-elementary lattices

We end this section by introducing p-elementary lattices. As highlighted in Lemma 2.1.14(4) and explored further in Section 3.3, these lattices play a key role in constructing and understanding the automorphism groups of K3 surfaces. This is particularly significant for the classification of K3 surfaces up to isometry.

Definition 2.1.21 (p-elementary lattice)**.** Let L be a lattice and p a prime number. We say that L is *p-elementary* if its discriminant group $A(L)$ is isomorphic to $(\mathbb{Z}_p)^{l(A(L))}$.

Example 2.1.22. Fix a prime p. The lattices $\langle 1 \rangle$ and $\langle p \rangle$ are the only lattices of rank one which are p-elementary. The lattice $U(p)$ is also a p-elementary lattice, as its discriminant group is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ (see Example 2.1.4). Similarly, U is p-elementary for any p prime, since it has trivial discriminant group.

In general, the lattices U and $U(p)$ are not the only p-elementary lattices of rank two. We highlight the following two examples.

Example 2.1.23. The even non-degenerate lattice $\langle 2 \rangle \oplus \langle -2 \rangle$ has rank two and its intersection matrix is given by

$$\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}.$$

It is a 2-elementary lattice since its discriminant group is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Example 2.1.24. The even non-degenerate lattice H_5 of rank two with intersection matrix

$$\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix},$$

is a 5-elementary lattice since its discriminant group $A(H_5)$ is isomorphic to $\mathbb{Z}_5$.

Remark 2.1.25. Let L be a unimodular lattice and $\varphi \in O(L)$ an isometry of order p. The invariant lattice L^φ and co-invariant lattice L_φ are both p-elementary, by Lemma 2.1.14 and Proposition 2.1.8. In particular, if $\mathrm{rank}(L^\varphi) = \mathrm{rank}(L)$ or $\mathrm{rank}(L_\varphi) = \mathrm{rank}(L)$, L is also a p-elementary lattice.

In this thesis, we are particularly interested in p-elementary hyperbolic lattices. The 2-elementary lattices were classified by Nikulin in [Nik83, §4]. Here, we recall this classification for 2-elementary hyperbolic lattices of rank ≤ 2.

Lemma 2.1. *Let L be a 2-elementary hyperbolic lattice of rank ≤ 2. If L has rank one, L is isomorphic to $\langle 2\rangle$. If L has rank two, it is isomorphic to one of the following lattices: U, $U(2)$ or $\langle 2\rangle \oplus \langle -2\rangle$.*

2.2 Background on algebraic geometry

In this section we give some general definitions and results in algebraic and birational geometry. All varieties are assumed to be projective and irreducible over the field of complex numbers $\mathbb{C}$. By a curve we always mean an irreducible and reduced curve.

2.2.1 Divisors, 1-cycles and intersection numbers

We start by recalling briefly basic definitions about divisors and intersection numbers. We refer to [Laz04, Section 1.1] and [Deb01, Chapter 1] for more details.

Let X be a normal projective variety, denote by $\mathcal{K}_X$ the sheaf of rational functions and by $\mathcal{O}_X$ the structure sheaf of X. A *Weil divisor* on X is a formal finite sum of subvarieties of codimension 1 (called *prime divisors*) with integer coefficients. These divisors form an abelian group, denoted by $\mathrm{WDiv}(X)$.

$$\mathrm{WDiv}(X) = \left\{ \sum_{i=1}^{m} a_i Y_i | m \in \mathbb{N},\ a_i \in \mathbb{Z},\ Y_i \subset X \text{ subv. of codim. 1, for } i = 1, ..., m \right\}.$$

If $D = \sum a_i Y_i$ is a Weil divisor, we call D *effective* if $a_i \geq 0$ for every i. Given a rational function $f \in H^0(X, \mathcal{K}_X^*)$, for every subvariety of codimension 1 Y we associate the integer $\nu_f(Y)$ as follows: $\nu_f(Y) = k > 0$ if f vanishes on Y to the order k; $\nu_f(Y) = -k < 0$ if f has a pole of order k on Y, and $\nu_f(Y) = 0$ otherwise. $\nu_f(Y)$ is called *multiplicity* of f at Y. Since $\nu_f(Y) = 0$ for all but finitely many Y, we define

$$div(f) = \sum_{Y \text{ prime divisor}} \nu_f(Y) Y \quad \in \mathrm{WDiv}(X).$$

The divisors obtained in this way are called *principal divisors*. If f and g are rational functions of X, then $div(fg) = div(f) + div(g)$. It follows that principal divisors form a subgroup $\mathrm{PDiv}(X)$ of $\mathrm{WDiv}(X)$.

Two divisors D, D' on X are *linearly equivalent* is $D - D' \in \mathrm{PDiv}(X)$ is a principal divisor. We represent this relation as $D \sim D'$. The quotient of $\mathrm{WDiv}(X)$ by the subgroup of principal divisors is denoted by $\mathrm{Cl}(X)$ and is called *divisor class group.*

A *Cartier divisor* D is defined as a global section of the sheaf $\mathcal{K}_X^*/\mathcal{O}_X^*$, i.e., D is given by a collection of pairs (U_i, f_i) where (U_i) is an open cover of X, f_i is an invertible element of $\mathcal{K}_X(U_i)$ such that f_i/f_j is in $\mathcal{O}_X^*(U_i \cap U_j)$. Thus, a Cartier divisor is a locally principal Weil divisor. We denote by $\mathrm{CDiv} \subset \mathrm{WDiv}$ the group of Cartier divisors on X. Clearly, it contains all principal divisors.

Remark 2.2.1. On a smooth variety X, the notions of a Weil divisor and a Cartier divisor are equivalent, $\mathrm{WDiv}(X) = \mathrm{CDiv}$.

Let $\pi \colon Y \to X$ be a surjective morphism between varieties and D be a Cartier divisor on X given by (U_i, f_i). The *pullback* π^*D is a Cartier divisor on Y given by $(\pi^{-1}(U_i), f_i \circ \pi)$. If π is not surjective, the pullback of Cartier divisor is not well-defined in general. However, its class under linear equivalence is well-defined. For instance, if we consider a subvariety $Y \subset X$ and the inclusion map $\iota \colon Y \hookrightarrow X$, the pullback π^*D of a Cartier divisor is exactly the restriction $D|_Y$ of D to Y and it makes sense if Y is not contained in the support of D.

For any Cartier divisor D, given by (U_i, f_i), we associate to it a subsheaf $\mathcal{O}_X(D)$ of $\mathcal{K}_X$, namely the $\mathcal{O}_X$-module locally generated by $1/f_i$. Any invertible sheaf is obtain in this way and moreover, two divisors D and D' are linearly equivalent if and only if the corresponding sheaves $\mathcal{O}_X(D)$ and $\mathcal{O}_X(D')$ are isomorphic. A similar association can be constructed, where Weil divisors correspond to rank one reflexive sheaves.

The *Picard group* $\mathrm{Pic}(X)$ of X is the group of isomorphism classes of line bundles over X. Using the identification of $\mathrm{Pic}(X)$ with the group of isomorphism classes of invertible sheaves, we have the following identification

$$\mathrm{CDiv}(X)/\,\mathrm{PDiv}(X) \cong \mathrm{Pic}(X).$$

Definition 2.2.2 (Canonical divisor)**.** Let X be a smooth projective variety of dimension n. Recall that Ω_X^p denotes the sheaf of p-rational forms and the canonical sheaf $\omega_X = \Omega_X^n$ is the invertible sheaf of n-rational form. The *canonical divisor*, denoted by K_X, is defined as the equivalence class of the Cartier divisors such that $\omega_X \cong \mathcal{O}_X(K_X)$.

When X is a normal singular projective variety, the non-singular locus $U \subset X$ is an open set whose complement has codimension ≥ 2. Thus, U is a smooth quasi-projective variety and its canonical divisor K_U can be obtained as before. We write $K_U = \sum a_i D_i$, where $D_i \in \mathrm{WDiv}(U)$ and $a_i \in \mathbb{Z}$. Hence, the canonical divisor on X can be defined as

$$K_X := \sum a_i \overline{D}_i,$$

where $\overline{D}_i$ is the closure of D_i in X. Since $\mathrm{codim}(X \setminus U) \geq 2$, K_X is the unique Weil divisor on X such that $K_X|_U = K_U$.

A *1-cycle* C on X is a formal finite sum $\sum a_i C_i$ of curves C_i with integer coefficients a_i. When $a_i \geq 0$ for any coefficient in the sum, C is said to be an *effective* cycle.

We are interested in the intersection number of a Cartier divisor with a 1-cycle, and more generally, the intersection of r Cartier divisors $D_1, \dots, D_r$, where $r \geq \dim X$.

Definition/Proposition 2.2.3 (Intersection number)**.** Let X be a normal projective variety of dimension n. Let $D_1, \dots, D_r \in \mathrm{CDiv}(X)$, with $r \geq \dim X$.

1. The intersection number $D_1 \cdot D_2 \cdots D_r$ is an integer number that is zero whenever $r > n$. The map $(D_1, \ldots, D_n) \mapsto D_1 \cdots D_n$ is multilinear, symmetric and take integral values.
2. The intersection number $D_1 \cdots D_n$ depends only on the linear equivalence classes of the D_i's.
3. Let $Y \subset X$ be a subvariety of dimension s. Then $D_1 \cdots D_s \cdot Y = D_1|_Y \cdots D_s|_Y$, where D_i is replaced, if necessary, with a linear equivalent Cartier divisor D_i' which does not contains Y in its support.
4. Let C be a curve and $D \in \mathrm{CDiv}(X)$. Then the intersection of D with the curve C is defined as $D \cdot C := \deg(\eta^* \mathcal{O}_X(D)|_C)$, where $\eta \colon \tilde{C} \to C$ is the normalization of C. We extend the intersection of a divisor with a 1-cycle by linearity.

When $D_1, D_2, \ldots, D_n$ are subvarieties of codimension 1 meeting properly in a finite number of points, $D_1 \cdots D_n$ is the number of points $D_1 \cap \cdots \cap D_n$ counted with multiplicity. In the intersection theory, the intersection of r Cartier divisors when $r \leq \dim X$, or more general, the intersection of any two subvarieties on X can be defined. Intuitively, if A and B are subvarieties of X, the intersection $A \cdot B$ of A with B is a subvariety of codimension $\mathrm{codim}(A) + \mathrm{codim}(B)$ (see [EH16, Chapter 1]).

Let $\pi \colon Y \to X$ be a proper morphism beween varieties and C be a curve on Y. We define $\pi_* C$ as follows

$$\pi_* C = \begin{cases} 0, & \text{if } C \text{ is contracted by } \pi; \\ d\pi(C), & \text{otherwise,} \end{cases}$$

where d is the degree of the restriction $C \to \pi(C)$ of π to C.

Proposition 2.2.4 (Projection formula)**.** Let $\pi \colon Y \to X$ be a morphism.

1. Assume that π is generically finite, proper, surjective and let $D_1, \ldots, D_r \in \mathrm{CDiv}(X)$, with $r \geq \dim X$. Then $\pi^* D_1 \cdots \pi^* D_r = \deg(\pi) D_1 \cdots D_r$.
2. Let $D \in CDiv(X)$ and C be a curve on Y. Then $\pi^* D \cdot C = D \cdot \pi_* C$.

The intersection number is not well-defined for Weil divisors in general. However, when a multiple of a Weil divisor is Cartier, we can extended naturally this notion.

Definition 2.2.5. A Weil divisor D on a projective variety X is called $\mathbb{Q}$-*Cartier* if there exists a positive integer m such that mD is a Cartier divisor. We say that X is $\mathbb{Q}$-*factorial* if every Weil divisor on X is $\mathbb{Q}$-Cartier.

Thus, for $\mathbb{Q}$-Cartier divisors $D_1, \ldots, D_n$ on a projective variety X of dimension n such that $D_i' = m_i D_i$ is Cartier, for some $m_i \in \mathbb{N}$, the intersection product is defined by

$$D_1 \cdots D_n = \frac{1}{m_1 \cdots m_n} D_1' \cdots D_n' \in \mathbb{Q}.$$

Similarly, we define the intersection product of a $\mathbb{Q}$-Cartier divisor with a curve.

We conclude this subsection with the following result, which will be useful in Section 4.3. It relates the Picard group of a smooth three-dimensional variety to the Picard group of its blowup along a smooth curve, as well as their intersection number.

Proposition 2.2.6 ([IP99, Lemma 2.2.14])**.** Let X be a smooth threefold and C be a smooth curve on X. Denote by $\pi\colon X' \to X$ the blowup of X along C, by E the exceptional divisor and e a fiber of $\pi|_E$. Then X' is smooth and

1. $\operatorname{Pic}(X') = \pi^* \operatorname{Pic}(X) \oplus \mathbb{Z}E$.
2. $E \cdot e = -1$ and $E^3 = -\deg(N_{C/X})$.
3. $E \cdot \pi^* D = (D \cdot C)e$ and $\pi^* D \cdot e = 0$, for any divisor D on X.
4. $E \cdot \pi^* C' = \pi^* C' \cdot e = 0$, for any 1-cycle C' on X.

2.2.2 Cone of curves and divisors

In this subsection, we introduce numerical equivalence, defined through the intersection number. Additionally, we discuss certain $\mathbb{R}$-vector spaces and cones within them. These cones serve as fundamental tools for analyzing how divisors and curves influence the geometry of a variety.

Definition 2.2.7 (Numerical equivalence)**.** Let X be a variety.

1. Two Cartier divisors D and D' on X are said to be *numerically equivalent*, written $D \equiv D'$, if $D \cdot C = D' \cdot C$ for any 1-cycle C on X. The group of numerical equivalence classes of Cartier divisors is denoted by $N^1(X)$ and called the *Néron-Severi group*.
2. Two 1-cycles C and C' on X are said to be *numerically equivalent*, written $C \equiv C'$, if $D \cdot C = D \cdot C'$ for any Cartier divisor D on X. We denote by $N_1(X)$ the group of numerical equivalence classes of 1-cycles.
3. We define the $\mathbb{F}$-vectorial spaces $N^1(X)_{\mathbb{F}} := N^1(X) \otimes_{\mathbb{Z}} \mathbb{F}$ and $N_1(X)_{\mathbb{F}} := N_1(X) \otimes_{\mathbb{Z}} \mathbb{F}$, for $\mathbb{F} = \mathbb{Q}$ or $\mathbb{F} = \mathbb{R}$.

By the Theorem of the base of Néron-Severi, the Néron-Severi group $N^1(X)$ is a free abelian group of finite rank $\rho(X)$, called the *Picard number* of X. The intersection number of Definition/Proposition 2.2.3(4) induces a perfect pairing $N^1(X)_{\mathbb{R}} \times N_1(X)_{\mathbb{R}} \to \mathbb{R}$ making $N^1(X)_{\mathbb{R}}$ and $N_1(X)_{\mathbb{R}}$ dual spaces.

Example 2.2.8. Assume S is a smooth surface. On such a surface, linear and numerical equivalences are the same. This means that if two divisors are linearly equivalent, they are also numerically equivalent, and vice-versa. Thus $\operatorname{Pic}(S) \cong N^1(S)$. Moreover, since subvarieties of S of codimension one are curves, divisors and 1-cycles coincide. Therefore, we have an identification of $N^1(S) = N_1(S)$ and so the intersection number induces a non-degenerate symmetric bilinear form $N^1(S)_{\mathbb{R}} \times N^1(S)_{\mathbb{R}} \to \mathbb{R}$. This bilinear form satisfies the following. We refer to [Har77, Chapter V, Theorem 1.9] for a proof.

Theorem 2.2.9 (Hodge index theorem)**.** Let S be a smooth surface. Then the non-degenerate symmetric bilinear form $N^1(S)_{\mathbb{R}} \times N^1(S)_{\mathbb{R}} \to \mathbb{R}$ has signature $(1, \rho(S) - 1)$, i.e., it can be diagonalized with entries $(1, -1, \ldots, -1)$.

Now, for a given variety X, we provide the definition of relevant cones in the spaces $N^1(X)$ and $N_1(X)$.

Definition 2.2.10 (Nef divisor and nef cone)**.** A Cartier divisor D is called *nef* if $D \cdot C \geq 0$ for every effective 1-cycle C on X. The *nef cone* $\operatorname{Nef}(X) \subset N^1(X)_{\mathbb{R}}$ is the closed convex cone generated by nef divisors.

Next, we define the Mori cone, which is fundamental to the Minimal Model Program (MMP), as the steps of the program (extremal contractions and flips) are determined by the extremal rays of the Mori cone (see Section 2.2.3).

Definition 2.2.11 (Mori cone)**.** The *cone of curves* $\operatorname{NE}(X) \subset N_1(X)$ is the convex cone generated by classes of effective 1-cycles, i.e.,

$$\operatorname{NE}(X) = \left\{ \sum n_i [C_i] | C_i \text{ is a curve and } n_i \geq 0 \right\}.$$

Its closure $\overline{\operatorname{NE}}(X)$ is called the *Mori cone*.

By definition, the dual cone of the Mori cone $\overline{\operatorname{NE}}(X)$ is the nef cone $\operatorname{Nef}(X)$. A subcone $N \subset \overline{\operatorname{NE}}(X)$ is called *extremal* if for any two elements $u, v \in \overline{\operatorname{NE}}(X)$ with $u + v \in N$, we have that $u, v \in N$. If N is an extremal face of dimension 1, we say that N is an *extremal ray*. In general, $\overline{\operatorname{NE}}(X)$ may be round and an extremal ray may not be generated by a class of a curve.

An extremal ray $R \subset \overline{\operatorname{NE}}(X)$ is called K_X*-negative*, K_X*-positive*, or K_X*-trivial* if $K_X \cdot \alpha < 0$, $K_X \cdot \alpha > 0$, or $K_X \cdot \alpha = 0$, respectively, for all $\alpha \in R \setminus \{0\}$. By a slight abuse of notation, we write $K_X \cdot R < 0$ to indicate that $K_X \cdot \alpha < 0$ for all $\alpha \in R \setminus \{0\}$. The same convention applies for $K_X \cdot \alpha > 0$ and $K_X \cdot \alpha = 0$.

Definition 2.2.12 (Relative cones)**.** Let $\pi \colon X \to Y$ be a morphism between projective varieties. We denote by $N_1(X/Y)$ the subspace of $N_1(X)$ generated by classes of curves contracted by π. Two Cartier divisors D and D' on X are *numerically equivalent over* Y, written $D \equiv_Y D'$, if $D \cdot C = D' \cdot C$ for every curve $C \in N_1(X/Y)$. The quotient of $N^1(X)$ by this relation is denoted by $N^1(X/Y)$. The *relative Picard number* $\rho(X/Y)$ is the rank of the free abelian group $N^1(X/Y)$. The *relative cone of curves* $\operatorname{NE}(X/Y)$ is defined as the cone $\operatorname{NE}(X) \cap N_1(X/Y)_{\mathbb{R}}$ in $N_1(X/Y)_{\mathbb{R}}$, and so in $N_1(X)_{\mathbb{R}}$ it is generated by classes of curves contracted by π.

Note that $\operatorname{NE}(X/Y) := \operatorname{NE}(X) \cap \ker(\pi_*)$. For convention we also denote the relative cone of curves by $\operatorname{NE}(\pi)$.

Proposition 2.2.13 ([Deb01, Proposition 1.14])**.** Let $\pi \colon X \to Y$ be a morphism between normal projective varieties. Then,

1. The class of a curve C lies in $\operatorname{NE}(\pi)$ if and only if C is contracted by π.
2. $\operatorname{NE}(\pi)$ is an extremal subcone of $\operatorname{NE}(X)$.
3. If $\pi_* \mathcal{O}_X = \mathcal{O}_Y$ (equivalently, π has connected fibers), then π is uniquely determined, up to isomorphism, by the extremal subcone $\operatorname{NE}(\pi)$.

We name morphisms satisfying the condition (3) of Proposition 2.2.13.

Definition 2.2.14. A *contraction* $\pi \colon X \to Y$ is a surjective morphism with connected fibers between normal projective varieties. By Proposition 2.2.13, it is uniquely associated with the extremal subcone $\operatorname{NE}(\pi)$. When $\operatorname{NE}(\pi)$ is an extremal ray, π is called an *extremal contraction*.

We remark the following. Since the intersection number depends on the linear equivalence class of a Cartier divisor, it follows that linear equivalence implies numerical equivalence and so there is a natural surjective map $\mathrm{Pic}(X) \to N^1(X)$. Next, we will see that some geometric properties of divisors depend only on their numerical class.

Definition 2.2.15 (Ample divisor)**.** Let D be a Cartier divisor on X. D is called *very ample* if there exists and embedding $f\colon X \hookrightarrow \mathbb{P}^N$ such that $\mathcal{O}_X(D) = f^*\mathcal{O}_{\mathbb{P}^N}(1)$, i.e., D is the restriction of a hyperplane of $\mathbb{P}^N$ to X, under the embedding f. D is called *ample* if there exists a positive integer m such that mD is very ample.

Proposition 2.2.16 (Kleiman's ampleness criterion)**.** A Cartier divisor D on a projective variety X is ample if and only if $D \cdot C > 0$ for any $C \in \overline{\mathrm{NE}}(X) \setminus \{0\}$.

Definition 2.2.17 (Ample cone)**.** The *ample cone* $\mathrm{Amp}(X)$ of a projective variety X is the convex subcone of $N^1(X)_{\mathbb{R}}$ generated by classes of ample divisors.

For a Cartier divisor D, we associate the *complete linear system* $\mathbb{P}(H^0(X, \mathcal{O}_X(D)))$, or equivalently the space $|D|$ of effective divisors that are linearly equivalent to D. The *base locus* $Bs|D|$ is the set of points of X where every section $s \in H^0(X, \mathcal{O}_X(D))$ vanishes. If $H^0(X, \mathcal{O}_X(D))$ is not trivial, the linear system $|D|$ induces a rational map

$$\varphi_{|D|}\colon X \dashrightarrow \mathbb{P} = \mathbb{P}(H^0(X, \mathcal{O}_X(D))^\vee),$$

which is defined in the complement of $Bs|D|$. The map $\varphi_{|D|}$ is a morphism exactly when $Bs|D| = \varnothing$, in this case, we say that $|D|$ is *base point free.*

Definition 2.2.18. Let D be a Cartier divisor on a projective variety X.

1. D is called *big* if there is a constant $A > 0$ such that $h^0(X, \mathcal{O}_X(mD)) \geq Am^{\dim X}$ for $m \gg 0$.
2. D is called *semiample* if there exists some multiple mD which is base point free.

We end this subsection by introducing the notion of $\mathbb{Q}$-divisors.

Definition 2.2.19 ($\mathbb{Q}$-divisors)**.** A $\mathbb{Q}$-*divisor* on a projective variety X is a $\mathbb{Q}$-linear combination of subvarieties of codimension 1. A $\mathbb{Q}$-divisor D is said to be $\mathbb{Q}$-Cartier if some integer multiple of D is a Cartier divisor. Two $\mathbb{Q}$-divisors D and D' are $\mathbb{Q}$-*linearly equivalent* if there exist a integer $m > 0$ such that both mD and mD' are Cartier divisor which are linearly equivalent. We denote this relation as $D \sim_{\mathbb{Q}} D'$.

Remark 2.2.20. The notions of intersection product, ampleness, nefness, bigness and semiampleness can be extended naturally to $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors.

2.2.3 Minimal Model Program

In this section, we introduce the Minimal model program, briefly outline how it works and recall the key definitions and results derived from it that are essential for the development of this thesis. For further details, we refer to [KMM87], [KM98] and [Mat02].

The Minimal Model Program (MMP) is a program for the construction of "simplest" representatives of each birational class of mildly singular projective varieties. The MMP plays a crucial role in the birational classification of projective varieties, as it reduces the problem to studying only the outputs of the program. In the one-dimensional case, it is well-known that any complex algebraic curve is birational to a unique smooth projective curve. Since smooth projective curves correspond to compact Riemann surfaces, they are fully classified by their genus.

The classical 2-dimensional version of the MMP was provided by the Italian school in their work on the birational classification of projective surfaces. The procedure is as follows. Given a projective surface S', it is birationally equivalent to a smooth projective surface S. Thus, we perform an algorithm from S, whose main ingredient is Castelnuovo's contractibility criterion. This criterion states that whenever a smooth projective surface S contains a (-1)-curve C, i.e., a rational curve $C \cong \mathbb{P}^1$ with $C^2 = -1$, S can be blown down to a smooth surface with exceptional curve C. This is called *contraction* of C. Repeating this process a finite number of times, we end up with a smooth projective surface $\tilde{S}$ without (-1)-curves. Surfaces satisfying this last property are called *minimal surfaces*. For instance, the minimal surfaces in the birational class of rational surfaces are $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and the Hizerbruch surfaces $\mathbb{F}_n$, $n \geq 2$.

It can be verified that any (-1)-curve C on a smooth projective surface S generates a K_S-negative extremal ray $R \subset \overline{\mathrm{NE}}(S)$, and the contraction of C corresponds to the contraction of the ray R. Furthermore, if a smooth surface S satisfies that K_S is nef, then S is immediately a minimal surface. This perspective is the foundation for extending the MMP to higher-dimensional varieties. In dimension three, it was fully established by Mori [Mor79], and more recently Birkar, Cascini, Hacon, and McKernan [BCHM10] achieved a major breakthrough in higher dimensions.

Given a smooth projective variety X, the first step in the modern MMP is to ask whether K_X is nef. If it is, X is a *minimal model* and the program stops. If not, the next task is to find a K_X-negative extremal ray R which can be contracted. However, complications arise. The first issue is that singularities become unavoidable because the contraction of a K_X-negative extremal ray may result in a singular target variety. Consequently, a whole theory of singularities was developed in the context of the MMP. The minimal requirement for varieties in this framework is that they have a $\mathbb{Q}$-Cartier canonical divisor, so the question of nefness is well-defined. To conclude, we now define the smallest class of singularities that appear in the MMP.

Definition 2.2.21 (Terminal/canonical singularities)**.** Let X be a normal $\mathbb{Q}$-factorial variety and $f\colon Y \to X$ be a log resolution of X, i.e., Y is a smooth projective variety, f is a birational morphism whose exceptional locus is $\mathrm{Exc}(f) = \sum E_i$, with E_i prime divisors on Y, and the divisor $\mathrm{Exc}(f)$ has simple normal crossing support. Write

$$K_Y \sim_{\mathbb{Q}} f^*K_X + \sum a_i E_i.$$

We say that X has *terminal singularities* if $a_i > 0$ for all i, and *canonical singularities* if $a_i \geq 0$.

It turns out that this condition does not depend on the choice of log resolution $f\colon Y \to X$. Each *discrepancy* $a_i = a(E_i)$ is a rational number depending uniquely on the valuation ν_{E_i} on $\mathbb{C}(X)$ associated to E_i. We refer to [KM98, Definition 2.25] for the notion of discrepancy. Additionally, we sometimes say that X "is" terminal (resp. canonical) to indicate that X has terminal singularities (resp. canonical singularities).

Terminal singularities are sufficient to ensure that the MMP is well-defined; in other words, if we start the MMP with a terminal variety, the variety remains terminal at each step. Thus, the issue of singularities is resolved. Each contraction of a K_X-negative extremal ray falls into one of three categories: it is either a contraction to a lower-dimensional variety (*Mori fiber space*), a contraction whose exceptional locus has codimension one (*Divisorial contraction*), or a contraction whose exceptional locus has codimension at least two (*small contraction*). We formally define these three cases below.

Definition 2.2.22 (Mori fiber space)**.** A *Mori fiber space* is a normal $\mathbb{Q}$-factorial and terminal projective variety X together with a morphism $f\colon X \to B$ satisfying:

1. $f_*\mathcal{O}_X = \mathcal{O}_B$,
2. $-K_X$ if f-ample,
3. $\rho(X/B) = \rho(X) - \rho(B) = 1$,
4. $\dim X > \dim B$.

Recall that condition (1) is equivalent to requiring that f has connected fibers, as stated in Zariski Main Theorem (see [Har77, Corollary III.11.4]. Consequently, it is associated with an extremal ray $R \subset \overline{\mathrm{NE}}(X)$ such that $K_X \cdot R < 0$, by Proposition 2.2.13(3) and conditions (3) and (2). Finally, condition (4) guarantees that it is indeed an outcome of the MMP. We sometimes denote a Mori fiber space $X \to B$ as X/B.

Example 2.2.23 (Fano variety)**.** Let X be a smooth variety. We say that X is a *Fano variety* if its *anticanonical divisor* $-K_X$ is ample. Assume that X has Picard rank $\rho(X) = 1$. Thus, the morphism $X \to \mathrm{Spec}(\mathbb{C})$ endows X with the structure of a Mori fiber space. In particular, the projective spaces $\mathbb{P}^n$ are Mori fiber spaces.

Definition 2.2.24 (Divisorial contraction)**.** A *divisorial (extremal) contraction* is a birational contraction $f\colon X \to Z$ between $\mathbb{Q}$-factorial terminal varieties such that

1. $-K_X$ is f-ample,
2. $\rho(X/Z) = 1$,
3. the exceptional locus $\mathrm{Exc}(f)$ is a prime divisor of X

The divisor of X is contracted onto a subvariety of codimension ≥ 2 in Z, which is called the *center* of the divisorial contraction.

By definition of contraction (see Definition 2.2.14) and condition (2), f is associated with an extremal ray $R \subset \overline{\mathrm{NE}}(X)$. Condition (1) implies that $-K_X$ restricted to the fiber is ample, i.e., for any curve C contained in R, $-K_X \cdot C > 0$.

Definition 2.2.25 (Small contraction)**.** An extremal contraction $f\colon X \to Z$ from a $\mathbb{Q}$-factorial terminal variety X is *small* if the exceptional locus $\mathrm{Exc}(f)$ has codimension at least 2 in X.

The second new challenge in the higher dimensional MMP arises when the contraction of a K_X-negative extremal ray is a small contraction. More generally, when a small contraction $X \to Z$ contracts curves that intersect non-trivially with K_X, the canonical divisor K_Z of Z fails to be $\mathbb{Q}$-Cartier. The concept of a *flip*, introduced by Mori, is used to resolve this issue. We provide its definition, along with the definitions of flops and antiflips, as these notions will appear in Chapter 4. First, we define the following.

Definition 2.2.26 (Pseudo-isomorphism)**.** A birational map $f\colon X \dashrightarrow X'$ is called a *pseudo-isomorphism* if it is an isomorphism in codimension one, i.e., there exist open subsets $U \subset X$ and $V \subset X'$ such that $X \setminus U$ and $X' \setminus V$ have codimension > 1 and $f|_U : U \xrightarrow{\sim} V$. In this case we use the notation $f\colon X \dashrightarrow X'$.

Definition 2.2.27 (Flip, flop, antiflip)**.** Let $\chi\colon X \dashrightarrow X^+$ be a pseudo-isomorphism between $\mathbb{Q}$-factorial terminal varieties fitting into a commutative diagram

$$\begin{array}{ccc} X & \overset{\chi}{\dashrightarrow} & X^+, \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f^+} & \\ & Z & \end{array}$$

where $f\colon X \to Z$ and $f^+\colon X^+ \to Z$ are small contractions associated to extremal rays $R \subset \overline{\mathrm{NE}}(X)$ and $R^+ \subset \overline{\mathrm{NE}}(X^+)$. We say that

1. χ is a *flip* if $K_X \cdot R < 0$ and $K_{X^+} \cdot R^+ > 0$,
2. χ is an *antiflip* if $K_X \cdot R > 0$ and $K_{X^+} \cdot R^+ < 0$,
3. χ is a *flop* if $K_X \cdot R = 0 = K_{X^+} \cdot R^+$.

Remark 2.2.28. More generally, we can define a *D-flip or D-antiflip* for any $\mathbb{Q}$-Cartier divisor D by requiring that the extremal ray $R \subset \overline{\mathrm{NE}}(X)$ has a negative or positive intersection with D, respectively, and the extremal ray $R^+ \subset \overline{\mathrm{NE}}(X^+)$ has positive or negative intersection with D^+, respectively, where D^+ is the strict transform of D under χ. A *D-flop* is defined as a diagram as above that is both an ordinary flop and a D-flip, i.e., $K_X \cdot R = 0 = K_{X^+} \cdot R^+$, $(K_X + D) \cdot R < 0$ and $(K_{X^+} + D^+) \cdot R^+ > 0$.

The third problem concerns the existence of flips. This was proven in dimension 3 in [Mor88], and later for any dimension in [BCHM10]. The existence of a D-flip is equivalent to the finite generation of the $\mathcal{O}_Z$-algebra $A := \bigoplus_{m \geq 0} f_* \mathcal{O}_X(mD)$ which remains an open question for arbitrary D. However, it has been established for several significant cases. Given the existence of any D-flip, it is unique up to isomorphism and f^+ is precisely the morphism $\mathrm{Proj}_X(A) \to Z$.

Now, we are ready to outline how the modern MMP works. We start with a $\mathbb{Q}$-factorial terminal variety X and ask whether K_X is nef. If K_X is nef, the program stops and X is a minimal model. If K_X is not nef, we choose a K_X-negative extremal ray R and perform its contraction $X \to X'$. If the contraction is a Mori fiber space, the process terminates. If it is a divisorial contraction, X' remains a $\mathbb{Q}$-factorial terminal variety, so we replace X by X' and repeat the process. If the contraction $X \to X'$ is small, we perform the flip $X \dashrightarrow X^+$. Here, X^+ is again a $\mathbb{Q}$-factorial terminal variety, so we replace X by X^+ and continue. To conclude the program, it must be shown that the process eventually terminates. Each divisorial contraction $X \to X'$ reduces the Picard number by one, i.e., $\rho(X') = \rho(X) - 1$. Therefore, only a finite number of divisorial contractions can occur. The situation is different for flips since, for any flip $X \dashrightarrow X^+$, the Picard number is unchanged $\rho(X^+) = \rho(X)$. Thus, there is the possibility of an infinite sequence of flips. This leads to the fourth major problem in the MMP, known as *termination of flips*. Shokurov [Sho86] solved this problem for dimension 3, and Kawamata, Matsuda and Matsuky [KMM87] in dimension 4, but it remains open in higher dimensions. When the process ends, it does so under one of two conditions: either we obtain a *minimal model* or a *Mori fiber space*. In this thesis, we focus particularly on the former case.

We conclude this section by introducing the more general version of the MMP: the D-MMP, also known as the minimal model program for pairs (X, D). This version operates similarly to the standard MMP, with the key difference being that the role of K_X is replaced by $K_X + D$, as we see below.

A pair (X, D) always consists of a normal projective variety X and an effective $\mathbb{Q}$-divisor $D = \sum d_i D_i$ such that $K_X + D$ is $\mathbb{Q}$-Cartier. Analogous to Definition 2.2.21, we have the notion of singularities for pairs.

Definition 2.2.29. Let (X, D) be a pair and $f : Y \to X$ be a log resolution, i.e., a resolution of X such that $\operatorname{Supp}(f^{-1}(D) + \operatorname{Exc}(f))$ has pure dimension 1 and is simple normal crossing. Denote by E_i the exceptional divisors of f. Write

$$K_Y + f_*^{-1}D \sim_{\mathbb{Q}} f^*(K_X + D) + \sum a_i E_i.$$

We say that the pair (X, D) has

1. *terminal singularities* if $a_i > 0$ for all i,
2. *canonical singularities* if $a_i \geq 0$ for all i,
3. *Kawamata log terminal singularities* (klt) if $a_i > -1$ for all i, or
4. *log canonical singularities* (lc) if $a_i \geq -1$ for all i.

Similarly to Definition 2.2.21, each *discrepancy* $a_i = a(E_i, X, D)$ is a rational number that depends uniquely on the valuation ν_{E_i} of $\mathbb{C}(X)$ associated to E_i. Therefore, a_i does not depend on the choice of the log resolution f. For a precise definition of discrepancy, we refer to [KM98, Definition 2.25]. Furthermore, we sometimes say that the pair (X, D) "is" terminal (resp. canonical, klt, lc) to mean that (X, D) has terminal singularities (resp. canonical singularities, klt singularities, lc singularities).

Again, terminal singularities are sufficient for the D-MMP to be well-defined. A terminal pair (X, D) is a $K_X + D$-*minimal model* if $K_X + D$ is nef. If, however, we have a terminal pair (X, D) that is not a $K_X + D$-minimal model, we can contract a $K_X + D$-negative ray of the Mori cone $\overline{\mathrm{NE}}(X)$. This contraction can take one of the three forms: a $K_X + D$-Mori fiber space, a $K_X + D$-divisorial contraction or a $K_X + D$-small contraction. In all the three cases $-(K_X + D)$ is relatively ample with respect to the contraction. When the contraction is a $K_X + D$-small contraction, we perform a $K_X + D$-flip, as described in Remark 2.2.28.

Finally, the following result tells us that an extremal ray, for which the associated contraction results in a flip, flop, or antiflip, is generated by rational curves. Moreover, in the three-dimensional case, any curve that is contracted is isomorphic to $\mathbb{P}^1$.

Lemma 2.2.30 ([APZ24, Lemma 3.2])**.** Let $\varphi\colon X \dashrightarrow X^+$ be a pseudo-isomorphism as in Definition 2.2.27.

1. There exist divisors Δ and Δ^+ in $\mathrm{Pic}(X)_{\mathbb{Q}}$ and $\mathrm{Pic}(X^+)_{\mathbb{Q}}$ respectively, such that

$$(K_X + \Delta) \cdot R = (K_{X^+} + \Delta^+) \cdot R^+ = 0,$$

   and both (X, Δ) and (X^+, Δ^+) are klt.

2. The rays R and R^+ are both generated by rational curves.
3. If moreover $\dim(Z) = \dim(Z^+) = 3$, then every irreducible component of $\mathrm{Exc}(f)$ and $\mathrm{Exc}(f^+)$ is isomorphic to $\mathbb{P}^1$.

Chapter 3

K3 surfaces

K3 surfaces occupy a central place in the study of algebraic and complex geometry. The name "K3" was coined by Andre Weil in 1958, in honor of three eminent mathematicians: Kummer, Kähler, and Kodaira, as well as the K2 mountain in Karakoram. As smooth projective surfaces with trivial canonical bundle, K3 surfaces form one of the fundamental classes in Enriques' classification of complex surfaces. They belong to the class of surfaces with vanishing irregularity and share many properties with Calabi-Yau varieties, including their role as a 2-dimensional analog.

The study of K3 surfaces connects diverse areas of mathematics, including Hodge theory, lattice theory, and moduli spaces. A particularly powerful way to study K3 surfaces is through their lattice structure, specifically the second cohomology group $H^2(S, \mathbb{Z})$. The transcendental lattice of a K3 surface captures much of its complex structure, and automorphisms of K3 surfaces are closely linked to the symmetries of these lattices. The lattice structure also provides crucial information about the surface's Hodge theory, as it encodes the decomposition of cohomology into Hodge components. This deep connection between K3 surfaces and lattices allows for the application of tools from lattice theory and helps explain many of the geometric and topological properties of K3 surfaces.

This chapter explores the mathematical structures of K3 surfaces and their classification, with a particular focus on their relationship to lattices. For an extensive study of the topic we refer to [BPVdV84] and [Huy16].

3.1 Introduction

Definition 3.1.1. A *K3 surface* is a smooth surface S with irregularity $h^1(S, \mathcal{O}_S) = 0$ and trivial canonical divisor $K_S \sim 0$.

The triviality of K_S is equivalent to the existence of a unique (up-to scalar) nowhere vanishing global rational 2-form ω_S, i.e., $H^0(S, K_S) = \mathbb{C}\omega_S$.

The following are some classical examples of K3 surfaces.

Example 3.1.2 (Complete intersections)**.** A smooth complete intersection $S \subset \mathbb{P}^{n+2}$ of type $(d_1, \dots, d_n)$ is a K3 surface if and only if $\sum d_i = n+3$. Indeed, $H^1(S, \mathcal{O}_S) = 0$ by [Bea96, LemmaVIII.9], and $K_S \sim (\sum d_i - n - 3)H$, by the adjunction formula, where H is an hyperplane of $\mathbb{P}^{n+2}$. Thus, K_S is trivial, and so S is a K3 surface, if and only if $\sum d_i = n+3$. Moreover, we can assume that $2 = d_1 \leq d_2 \leq \cdots \leq d_n$. Indeed, if $d_1 = 1$, S is a complete intersection of type $(d_2, \dots, d_n)$ in $\mathbb{P}^{n+1}$. Thus, $2n \leq n+3$ and so $n \leq 3$. Therefore, we are left with the following cases:

1. $S \subset \mathbb{P}^3$ is a smooth quartic surface, or
2. $S \subset \mathbb{P}^4$ is a smooth complete intersection of a quadric and a cubic hypersurface, or
3. $S \subset \mathbb{P}^5$ is a smooth complete intersection of three quadric hypersurfaces.

Example 3.1.3 (Double cover of $\mathbb{P}^2$)**.** Let $\pi\colon S \to \mathbb{P}^2$ be a double cover branched along a smooth sextic curve $C \subset \mathbb{P}^2$. Denote by l a line in $\mathbb{P}^2$. Thus $\pi_*\mathcal{O}_S \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-3)$ and $K_S \sim \pi^*(K_{\mathbb{P}^2} + 3l)$, by the canonical bundle formula for branched coverings. This implies that $h^1(S, \mathcal{O}_S) = 0$ and $K_S \sim \mathcal{O}_S$ and so S is a K3 surface. Moreover, setting $A := \pi^* l$ we obtain that $A^2 = 2$.

For a K3 surface S, we have that $h^0(S, \mathcal{O}_S) = h^2(S, \mathcal{O}_S) = 1$ and $h^1(S, \mathcal{O}_S) = 0$ by definition and Serre duality. Thus $\chi(\mathcal{O}_S) = 2$.

The following proposition presents the well-known Riemann-Roch theorem in the specific context of K3 surfaces, along with its direct consequence, the arithmetic genus formula, which we state for curves that, in principle, may be singular, reducible, or non-reduced; in other words, for any effective divisor. Additionally, we include the renowned Noether's formula. For the general case, we refer the reader to [Bea96, Theorem I.2, I.4, I.5]. Recall that the *arithmetic genus* of an effective divisor D on S is defined as $p_a(D) := 1 - \chi(D, \mathcal{O}_D)$.

Proposition 3.1.4 (Formulas for K3 surfaces)**.** Let S be a K3 surface. Then the following holds.

1 (Riemann-Roch). Let D be a divisor on S, then

$$\chi(S, D) = h^0(S, D) - h^1(S, D) + h^0(S, -D) = 2 + \frac{1}{2}D^2.$$

2 (Arithmetic genus). Let D be an effective divisor on S, then

$$2p_a(D) - 2 = D^2.$$

3 (Noether's formula). The topological Euler-Poincaré characteristic of S and the Euler characteristic of $\mathcal{O}_S$ are related by

$$\chi_{top}(S) = 12\chi(\mathcal{O}_X) = 24.$$

We point out that when D is a curve, i.e., an irreducible and reduced effective divisor, the arithmetic genus is a non-negative integer: $p_a(D) \geq 0$. A direct consequence of the Riemann-Roch theorem and the Arithmetic genus formula is the following result.

Proposition 3.1.5. Let $D \nsim 0$ be a divisor on a K3 surface S. Then:

1. If $D^2 \geq -2$ then either D or $-D$ is linearly equivalent to an effective divisor on S.

2. If D is a curve, then $D^2 \geq -2$ with equality exactly when $D \cong \mathbb{P}^1$ is a rational curve.

3. If D is effective with $h^0(S, D) = 1$, then $D^2 \leq -2$ and every irreducible reduced component $C \leq D$ is a rational curve, i.e., $C \cong \mathbb{P}^1$ and $C^2 = -2$.

4. If D is nef and $D^2 > 0$, then $h^0(S, D) = \frac{1}{2}D^2 + 2$.

5. If $D = C$ is a smooth curve on S, then $-2 \leq 2g(C) - 2 = C^2$ and $h^0(S, C) = g(C) + 1$.

Proof. Let D be a non-trivial equivalent divisor on S. Riemann-Roch theorem asserts that

$$h^0(S, D) + h^0(S, -D) \geq h^0(S, D) - h^1(S, D) + h^0(S, -D) = 2 + \frac{1}{2}D^2.$$

Thus, if we assume that $D^2 \geq -2$, then $h^0(S, D) \geq 1$ or $h^0(S, -D) \geq 1$, and only one of them is non-zero. Otherwise, there would exist effective divisors $E \sim D$ and $F \sim -D$ such that $D \cdot H = E \cdot H > 0$ and $-D \cdot H = F \cdot H > 0$, for any ample divisor H on S (such H exists since S is projective). This leads to a contradiction. Hence, we obtain (1).

Now, assume D is a curve. From the arithmetic genus formula, we have that $D^2 = 2p_a(D) - 2 \geq -2$. Observe that equality holds when $p_a(D) = 0$. If we consider the normalization $\eta\colon \tilde{D} \to D$ of D, if follows that $\tilde{D}$ has geometric genus zero and the map is an isomorphism. Consequently, $D \cong \mathbb{P}^1$ is smooth. This proves (2).

To prove (3), note first that D is the unique effective divisor in the linear system $|D|$ and $h^0(S, -D) = 0$, since $h^0(S, D) = 1$. Using Riemann-Roch, we conclude that $1 \geq h^0(S, D) - h^1(S, D) = 2 + D^2/2$, which implies that $D^2 \leq -2$. Now, let $C \leq D$ be a curve in the support of D. By (2), $C^2 \geq -2$. Assume, for contradiction, that $C^2 \geq 0$. Applying Riemann-Roch again, we get that $h^0(S, C) \geq 2$. Consequently, there exists another curve $C' \sim C$. Replacing C with C' in D, we obtain an effective divisor $D' \neq D$ such that $D' \sim D$, leading a contradiction.

Now, assume that D is a nef divisor with $D^2 > 0$. By Riemman-Roch and the nefness assumption, $h^0(S, D) \geq 1$ and $h^0(S, -D) = 0$. Moreover, D is big (see Definition 2.2.18(1)) since $h^0(S, mD) \geq 2 + m^2D^2/2 \geq m^2$, for any integer $m > 0$. Thus, we apply the Kodaira-Ramanujan theorem (see [Huy16, Chapter 2, Theorem 1.8]) to obtain (4).

Finally, when C is a smooth curve, the arithmetic and geometric genus coincide, and so $-2 \leq 2g(C) - 2 = C^2$. Furthermore, from the fundamental exact sequence associated to C we have the exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(C) \to \mathcal{O}_C(C|_C) \to 0.$$

By the adjunction formula, $K_C \sim (K_S + C)|_C = C|_C$. Hence, from the long exact sequence in cohomology, we conclude that $h^0(S, C) \leq 1 + h^0(C, K_C) = 1 + g(C)$. On the other hand, combining the Riemann-Roch theorem and the artihmetic genus formula, we get $h^0(S, C) \geq 1 + g(C)$. Thus, (5) holds. □

Recall that if a divisor D is ample on S is, it satisfies that $D \cdot C > 0$ for any curve $C \subset S$. The following result provides a criterion to determine whether a divisor on a K3 surface is ample. We refer to [Har77, Chapter V, Theorem 1.10] and [Huy16, Chapter 8, Theorem 1.2 and Corollary 1.6] for a proof of it.

Proposition 3.1.6 (Nakai–Moishezon–Kleiman criterion)**.** A divisor D on a K3 surface S is ample if and only if $D^2 > 0$, $D \cdot H > 0$ for an ample divisor H, and $D \cdot C > 0$ for every rational curve C.

Given a nef line bundle $H \in \text{Pic}(S)$, the next theorem establishes necessary and sufficient conditions for H to be very ample. This result is a combination of results proven in [SD74], although the specific formulation presented here is from [Mor84a, Theorem 5].

Proposition 3.1.7. Let D be a nef divisor on a K3 surface S such that $D^2 \geq 4$. Then D is very ample if and only if the following three conditions hold:

1. There is no irreducible curve $E \subset S$ such that $E^2 = 0$ and $D \cdot E \in \{1, 2\}$.
2. There is no irreducible curve $E \subset S$ such that $E^2 = 2$, and $D \sim 2E$.
3. There is no irreducible curve $E \subset S$ such that $E^2 = -2$ and $E \cdot D = 0$.

We conclude this section by exploring some properties of linear systems on a K3 surface. Let D be a divisor on S and $|D|$ its complete linear system. Recall that the base locus $Bs|D|$ is the set of points on S for where every section $s \in H^0(S, D)$ vanishes. Since S is a surface, $Bs|D|$ contains subvarieties (not necessarily irreducible or reduced) of dimension zero or one. The *fixed part* of $|D|$ is defined as the largest effective divisor F such that $F \leq D'$ for every $D' \in |D|$. Consequently, the linear system $|D - F|$ has no fixed part. Setting $M = D - F$, the *mobile part* of $|D|$ is defined as the linear system $|M|$.

We collect the main facts about linear systems in the following proposition, which can be found in [SD74].

Proposition 3.1.8. Let D be an effective divisor on a K3 surface S with $D^2 \geq 0$. Write $|D| = |M| + F$, where $|M|$ and F are the mobile and fixed part, respectively. Then:

1. The mobile part M is nef with $M^2 \geq 0$, $M \cdot D \geq 0$ and $|M|$ is base point free.
2. The fixed part F is a sum of rational curves $\sum a_i C_i$, with $a_i \geq 0$. Moreover, for any $F' \leq F$, it holds that $h^0(S, F') = 1$.
3. If $M^2 > 0$, then $D^2 \leq M^2$ and a general element in $|M|$ is an irreducible curve.
4. If $M^2 = 0$, then $M \sim aE$, where E is an irreducible curve with $p_a(E) = 1$ and $a \geq 1$. Moreover, $h^0(S, M) = \frac{1}{2}M^2 + 1 + a$.
5. If D is nef with $D^2 > 0$, then either $F = 0$ or $D \sim aE + C$, where E is an irreducible curve with $p_a(E) = 1$, C is a rational curve and $a \geq 2$.

Proof. We begin by proving (1). Assume there exists a curve $C \subset S$ such that $M \cdot C < 0$. Then C is contained in every element of $|M|$ which implies that $|M|$ has fixed part. Thus, M is nef. Now, suppose $M^2 \leq -2$. Let C be an irreducible reduced component of M. By Proposition 3.1.5(3), C is a rational curve and $h^0(S, C) = 1$. Therefore, any element of $|M|$ contains C, which again contradicts the fact that $|M|$ has no fixed part. Hence, $M^2 \geq 0$. Finally, from [SD74, Corollary 3.2], $|M|$ is base point free.

The fixed part F satisfies that $h^0(S, F) = 1$, and any component $F' \leq F$ is fixed, so $h^0(S, F') = 1$. Thus, (2) follows from Proposition 3.1.5(3).

To prove the first assertion of (3), note that if $M^2 > 0$, then Proposition 3.1.5(4) implies $h^0(S, M) = 2 + M^2/2$. Consequently,

$$\frac{D^2}{2} + 2 \leq h^0(S, D) = h^0(S, M) = \frac{M^2}{2} + 2 \implies D^2 \leq M^2.$$

Moreover, the second assertion of (3) and (4) follow by [SD74, Proposition 2.6].

Now we prove (5) following the approach outlined in [Huy16, Chapter 2, Corollary 3.15]. Assume that D is big and nef, i.e., D is nef and $D^2 > 0$. We separate into the two cases: $M^2 > 0$ ot $M^2 = 0$. Suppose $M^2 > 0$. Then M is nef and big. Thus, $h^0(S, D) = h^0(S, M)$ and $h^1(S, D) = h^1(S, M) = 0$, so $D^2 = M^2$. This implies $D^2 = (M + F)^2 = M^2 + 2M \cdot F + F^2$, and $0 = 2M \cdot F + F^2$. Since D is nef, $0 < D \cdot F = M \cdot F + F^2$, which implies $M \cdot F = 0 = F^2$. This is a contradiction, as $F^2 \leq -2$ by Riemann-Roch and the fact that $h^0(S, F) = 1$. Thus, we conclude that $F = 0$. Suppose now that $F \neq 0$, or equivalent, $M^2 = 0$. Then there exists an irreducible curve E, such that $h^0(S, E) = 1$, and an integer $a > 0$, such that $M \sim aE$. Note that $a > 1$; otherwise, $h^1(S, M) = h^1(S, E) = 0$ and $2 < h^0(S, D) = h^0(S, M) = h^0(S, E) = 1$, which is absurd.

Thus, $a > 1$. Observe that $0 < D^2 = (M + F)^2 = 2M \cdot F + F^2$. Since $F^2 \leq -2$, we deduce that $M \cdot F > 0$. Consequently, $C \cdot E > 0$ for at least one irreducible and reduced component $C \cong \mathbb{P}^1$. In particular, $M \cdot C \geq 2$. Therefore, $(M + C)^2 = 2M \cdot C - C^2 > 0$ and $(M + C) \cdot C = M \cdot C + C^2 \geq 0$. This implies $M + C$ is a big and nef divisor. Using similar arguments as in the case $M^2 > 0$, for the decomposition $D = (M + C) + F'$, we conclude $F' = 0$ and so $F = C$. ☐

3.2 Hodge and lattice structures

Let S be a K3 surface. For $p, q \in \{0, 1, 2\}$, we define the Hodge numbers as $h^{p,q} = \dim_{\mathbb{C}} H^q(S, \Omega_S^p)$, where Ω_S^p is the bundle of regular p-forms on S. Then the complex structure on $H^i(S, \mathbb{C})$, is given by

$$H^i(S, \mathbb{Z}) \otimes \mathbb{C} \cong H^i(S, \mathbb{C}) = \bigoplus_{p+q=i} H^{p,q}(S),$$

where $H^{p,q}(S)$ denotes $H^q(S, \Omega_S^p)$. By definition and Serre duality we have that $h^{0,0} = h^{2,0} = h^{0,2} = h^{2,2} = 0$. By the Noether's formula (see 3.1.4(3)) and the Poincare duality we have that the Euler characteristic of S is $\chi_{\text{top}} = 12\chi(\mathcal{O}_S) = 24 = 2b_0(S) + 2b_1(S) + b_2(S)$, where $b_i(S) = \dim_{\mathbb{R}} H^i(S, \mathbb{R}) = \dim_{\mathbb{C}} H^i(S, \mathbb{C})$ are the Betti numbers. Since $h^{1,0} = 0$, we get that $b_1 = 0$, $b_2 = 22 = h^{2,0} + h^{1,1} + h^{0,2}$ and so $h^{1,1} = 20$. Thus, the Hodge diamond of S is the following

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}.$$

The second cohomology group $H^2(S, \mathbb{Z})$ is a free $\mathbb{Z}$-module of rank 22, and endowed with the cup product it is an even unimodular lattice of signature $(3, 19)$, isometric to Λ_{K3} (see Example 2.1.7 and [BHPVdV04, Chapter

VIII, Proposition 3.3]). The Hodge structure of $H^2(S,\mathbb{Z})$,

$$H^2(S,\mathbb{Z}) \otimes \mathbb{C} \cong H^2(S,\mathbb{C}) = H^{2,0}(S) \oplus H^{1,1}(S) \oplus H^{0,2}(S),$$

is such that $H^{2,0}(S) = H^0(S,\Omega_S^2) \cong \mathbb{C}\omega_S$, $H^{0,2}(S) = \mathbb{C}\overline{\omega_S}$, and $H^{1,1}(S)$ is orthogonal to $H^{2,0}(S) \oplus H^{0,2}(S)$, under the extension of the cup product to $H^2(S,\mathbb{C})$. The exponential sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_S^* \longrightarrow 1,$$

induces a long exact sequence in cohomology

$$\cdots \longrightarrow H^1(S,\mathcal{O}_S) \longrightarrow H^1(S,\mathcal{O}_S^*) \longrightarrow H^2(S,\mathbb{Z}) \longrightarrow H^2(S,\mathcal{O}_S) \longrightarrow \cdots.$$

From the identification $\mathrm{Pic}(S) \cong H^1(S,\mathcal{O}_S^*)$ and the fact that $H^1(S,\mathcal{O}_S) = 0$, we have that $\mathrm{Pic}(S)$ is a subgroup of $H^2(S,\mathbb{Z})$ and the intersection number on S coincide with the cup product restricted to $\mathrm{Pic}(S)$. Moreover, we can identify naturally $H^2(S,\mathbb{Z})$ as a subspace of $H^2(S,\mathbb{C})$. The following proposition asserts that we can recover $\mathrm{Pic}(S)$ from the Hodge structure on $H^2(S,\mathbb{Z})$.

By $\langle x, y\rangle$, we indicate the cup product of x with y for every $x, y \in H^2(S,\mathbb{Z})$, and by $x \cdot y$, as usual, the intersection product of x with y for every $x, y \in \mathrm{Pic}(S)$.

Proposition 3.2.1 (Lefschetz theorem in $(1,1)$-classes, [BHPVdV04, Chapter IV, THeorem 2.13])**.** For a K3 surface we have

$$\mathrm{Pic}(S) \cong H^2(S,\mathbb{Z}) \cap H^{1,1}(S) = \{x \in H^2(S,\mathbb{Z}) | \langle x, \omega_S\rangle = 0\}.$$

Therefore, $\mathrm{Pic}(S)$ is an even lattice with Picard number $\rho(S) \le 20$ and signature $(1, \rho(S) - 1)$, by the Hodge index theorem (see Theorem 2.2.9). We define the lattice $\mathrm{T}(S)$ as the orthogonal complement $\mathrm{Pic}(S)^\perp$ of $\mathrm{Pic}(S)$ inside $H^2(S,\mathbb{Z})$. It has rank $22 - \rho(S)$ and signature $(2, 20 - \rho(S))$. The lattices $\mathrm{Pic}(S)$ and $\mathrm{T}(S)$ are called the *Picard* and *transcendental* lattice respectively, and they are primitive sublattices of $H^2(S,\mathbb{Z})$. The transcendental lattice $\mathrm{T}(S)$ has a sub-Hodge structure $\mathrm{T}(S)_\mathbb{C} = \mathrm{T}^{2,0}(S) \oplus \mathrm{T}^{1,1}(S) \oplus \mathrm{T}^{0,2}(S)$ of $H^2(S,\mathbb{Z})$, where $\mathrm{T}^{2,0}(S) = \mathbb{C}\omega_S$ by Proposition 3.2.1. Indeed, $\mathrm{T}(S)$ is the minimal lattice with sub-Hodge structure of $H^2(S,\mathbb{Z})$ with this property.

3.3 Automorphisms

The study of automorphisms of K3 surfaces reveals a profound interplay between geometry, topology, and arithmetic. The Global Torelli Theorem provides a foundational result, connecting the geometry of K3 surfaces to their Hodge structures and lattices, enabling a lattice-theoretic approach to understanding automorphisms. Automorphisms of K3 surfaces can be classified as symplectic or non-symplectic, depending on whether they preserve the holomorphic 2-form, and their analysis is closely tied to the lattice structure of the second cohomology group. This section is devoted to these topics, exploring the Torelli theorem, symplectic and non-symplectic automorphisms, and the role of lattice theory in classifying and analyzing the symmetries of K3 surfaces.

Definition 3.3.1. Let φ be an isometry of $H^2(S,\mathbb{Z})$.

1. φ is called a *Hodge isometry* if its $\mathbb{C}$-linear extension $\varphi_{\mathbb{C}}$ to $H^2(S,\mathbb{C})$ preserves the Hodge structure, i.e., $\varphi_{\mathbb{C}}(H^{2,0}(S)) = H^{2,0}(S)$.
2. φ is called *effective* if it sends an ample class to an ample class.

We denote by $\mathrm{Aut}(S)$ the group of automorphisms of a K3 surface S. Since the tangent bundle $\mathcal{T}_S$ is isomorphic to the contangent bundle Ω_S, it follows that $H^0(S,\mathcal{T}_S) = 0$. This implies that $\mathrm{Aut}(S)$ is a discrete group. The first result presented in this section asserts that any automorphism of a K3 surface S naturally induces a Hodge isometry of $H^2(S,\mathbb{Z})$. This is a first indication of the deep connection between the geometry of a K3 surface and the lattice and Hodge structure of its second cohomology group.

Proposition 3.3.2. Let $f \in \mathrm{Aut}(S)$ be an automorphism of S, then f^* is an effective Hodge isometry of $H^2(S,\mathbb{Z})$.

Proof. The pullback $f^*\colon H^2(S,\mathbb{Z}) \to H^2(S,\mathbb{Z})$ induced by f is an isometry of lattices. Since f is an isomorphism, the pullback of a 2-rational form of S is a 2-rational form. Hence, the extension $f^*_{\mathbb{C}}$ of f^* to $H^2(S,\mathbb{C})$ preserves $H^{2,0}(S)$. Moreover, $f^*_{\mathbb{C}}$ sends $H^{1,1}(S)$ and $H^{0,2}(S)$ to themselves because $f^*_{\mathbb{C}}$ commutes with the complex conjugation and preserves the bilinear form. Thus, f^* is a Hodge isometry.

Furthermore, f^* is effective, as the pullback f^*H of any given ample class H remains ample. Specifically, let m be a positive integer such that $|mH|$ determines an embedding $\varphi_{|mH|}\colon S \hookrightarrow \mathbb{P}^N$. Composing this embedding with f, the map $\varphi_{|mH|} \circ f$ is also an embedding of S in $\mathbb{P}^N$, given by the linear system $|mf^*H|$. □

We note that both lattices $\mathrm{Pic}(S)$ and $\mathrm{T}(S)$ are mapped to themselves by the isometry f^*, and more generally, for any Hodge isometry of $H^2(S,\mathbb{Z})$. This follows directly from Proposition 3.2.1. Moreover, the following theorem establishes that the converse of the preceding proposition is also true.

Theorem 3.3.3 (Global Torelli theorem)**.** Let S be a K3 surface and $\varphi\colon H^2(S,\mathbb{Z}) \longrightarrow H^2(S,\mathbb{Z})$ be a Hodge isometry sending an ample class to an ample class. Then there exists a unique automorphism f of S such that $f^* = \varphi$. In other words, we have the following one-to-one correspondence:

$$\mathrm{Aut}(S) \cong \{\varphi \in O(H^2(S,\mathbb{Z})) | \varphi \text{ is an effective Hodge isometry}\}.$$

The Global Torelli theorem allows us to investigate automorphisms of a K3 surface via isometries of its second cohomology group. By Proposition 2.1.11, the orthogonal group of $H^2(S,\mathbb{Z})$ can be characterized from isometries of $\mathrm{Pic}(S)$ and $\mathrm{T}(S)$ that glue appropriately.

Remark 3.3.4. For an effective isometry $\varphi \in O(H^2(S,\mathbb{Z}))$, the condition that φ preserves an ample class (or equivalent the ample cone $\mathrm{Amp}(S) \subset \mathrm{Pic}(S)_{\mathbb{R}}$) is determined by its restriction $\varphi_P = \varphi|_{\mathrm{Pic}(S)}$ to $\mathrm{Pic}(S)$. Meanwhile, the condition that φ preserves the Hodge structure of $H^2(S,\mathbb{Z})$ is determined by its restriction $\varphi_T = \varphi|_{\mathrm{T}(S)}$ to $\mathrm{T}(S)$. In fact, the $\mathbb{C}$-linear action on $H^{2,0}(S) = \mathrm{T}^{2,0}(S) = \mathbb{C}\omega_S$, induced by φ, takes the form $\varphi_{\mathbb{C}}\omega = \varphi_{T\mathbb{C}}\omega_S = \lambda\omega_S$, where $\lambda \in \mathbb{C}^*$. This action depends on the Hodge isometry φ_T of the transcendental lattice $\mathrm{T}(S)$. Thus, we arrive at the following conclusion:

$$\mathrm{Aut}(S) \cong \{(\varphi_P, \varphi_T) \in O(\mathrm{Pic}(S)) \times O(\mathrm{T}(S)) | \varphi_P \text{ preserves } \mathrm{Amp}(S),\ \varphi_T\omega_S = \lambda\omega_S \text{ with } \lambda \in \mathbb{C}^*, \text{ and } \overline{\varphi}_P = \overline{\varphi}_T\}.$$

Definition 3.3.5. Let S be a K3 surface.

1. An automorphism $f \in \mathrm{Aut}(S)$ is said to be *symplectic* if $f^*\omega_s = \omega_S$.
2. An automorphism $f \in \mathrm{Aut}(S)$ is said to be *non-symplectic* if $f^*\omega_s \neq \omega_S$.
3. A non-symplectic automorphism $f \in \mathrm{Aut}(S)$ is said to be *anti-symplectic* if $f^*\omega_s = -\omega_S$.

Determining whether an automorphism of a K3 surface is symplectic or anti-symplectic can be an easy task, as it is directly related to whether the action on the transcendental lattice is trivial up to a sign.

Proposition 3.3.6 ([Nik79, Theorem 3.1], [Huy16, Chapter 3, Lemma 3.3])**.** Let φ be a Hodge isometry of $H^2(S,\mathbb{Z})$. Its action on $H^{2,0}(S)$ is $\varphi_{\mathbb{C}}(\omega_S) = \omega_S$ (resp. $\varphi_{\mathbb{C}}(\omega_S) = -\omega_S$) if and only if its action on $\mathrm{T}(S)$ is $\varphi|_{\mathrm{T}(S)} = \mathrm{id}$ (resp. $\varphi|_{\mathrm{T}(S)} = -\mathrm{id}$).

Corollary 3.3.7. Let $f \in \mathrm{Aut}(S)$ be an automorphism of a K3 surface S. The following are equivalent:

1. f is symplectic (resp. anti-symplectic).
2. f acts on $H^{2,0}(S)$ as id, i.e., $f^*\omega_S = \omega_S$ (resp. f acts on $H^{2,0}(S)$ as $-\mathrm{id}$, i.e., $f^*\omega_S = -\omega_S$).
3. f acts on $\mathrm{T}(S)$ as id (resp. f acts on $\mathrm{T}(S)$ as $-\mathrm{id}$).

Let f be an automorphism of finite order n. The induced map f^* on $H^{2,0}(S) = \mathbb{C}\omega_S$ is a $\mathbb{C}$-linear automorphism and so, $f^*\omega_S = \lambda\omega_S$ for some $\lambda \in \mathbb{C}^*$. Since f has finite order n, it follows that $\omega_S = (f^*)^n\omega_S = \lambda^n\omega_S$, and thus λ is a n-th root of the unity. In particular, if f has order two, f is either symplectic or anti-symplectic. Automorphisms of finite order have been well understood by several authors from the lattice viewpoint. In particular, the invariant and co-invariant lattices $H^2(S,\mathbb{Z})^{f^*}$ and $H^2(S,\mathbb{Z})_{f^*}$ of the isometry f^* induced by an automorphism $f \in \mathrm{Aut}(S)$, as defined in Definition 2.1.13, play an important role.

Corollary 3.3.8. For every non-trivial automorphism f of finite order n, the following holds.

1. Both lattices $H^2(S,\mathbb{Z})^{f^*}$ and $H^2(S,\mathbb{Z})_{f^*}$ are non-trivial and primitively embedded in $H^2(S,\mathbb{Z})$.
2. f acts trivially on $A(H^2(S,\mathbb{Z})^{f^*}) \cong A(H^2(S,\mathbb{Z})_{f^*})$.
3. If f has finite prime order p, then both lattices $H^2(S,\mathbb{Z})^{f^*}$ and $H^2(S,\mathbb{Z})_{f^*}$ are p-elementary.
4. If f is symplectic, then $\mathrm{T}(S) \subset H^2(S,\mathbb{Z})^{f^*}$ and $H^2(S,\mathbb{Z})_{f^*} \subset \mathrm{Pic}(S)$.
5. If f is non-symplectic, then $H^2(S,\mathbb{Z})^{f^*} \subset \mathrm{Pic}(S)$ and $\mathrm{T}(S) \subset H^2(S,\mathbb{Z})_{f^*}$.

Proof. The fact that $H^2(S,\mathbb{Z})^{f^*}$ and $H^2(S,\mathbb{Z})_{f^*}$ are primitive sublattices of $H^2(S,\mathbb{Z})$ follows directly from Lemma 2.1.14(1), and the fact that f acts trivially on their discriminant groups follows from the definition of $H^2(S,\mathbb{Z})^{f^*}$, $A(H^2(S,\mathbb{Z})^{f^*})$ and Corollary 2.1.10. We prove now that $H^2(S,\mathbb{Z})^{f^*}$ is non-trivial. Indeed, let H be an ample class of $\mathrm{Pic}(S)$. It follows that $0 \neq H + f^*H + \cdots + (f^*)^{n-1}H$ is fixed by f^*. Moreover, if $H^2(S,\mathbb{Z})_{f^*}$ is trivial, this implies that f^* is the trivial isometry of $H^2(S,\mathbb{Z})$, by Lemma 2.1.14(3). Therefore, we obtain (1) and (2). Assertion (3) follows from Lemma 2.1.14(4).

Now, assuming that f is symplectic, we get that f^* acts trivially on the transcendental lattice. Thus, $\mathrm{T}(S) \subset H^2(S,\mathbb{Z})^{f^*}$, and by taking orthogonal complements on both sides, the containment changes, yielding $H^2(S,\mathbb{Z})_{f^*} \subset \mathrm{Pic}(S)$. This proves (4). Finally, we assume that f is non-symplectic to prove (5). Let $x \in H^2(S,\mathbb{Z})^{f^*}$. From $\langle x, \omega_S\rangle = \langle f^*x, f^*\omega_S\rangle = \lambda\langle x, \omega_S\rangle$, it follows that $(1-\lambda)\langle x, \omega_S\rangle = 0$. Since $\lambda \neq 1$, we deduce that $\langle x, \omega_S\rangle = 0$. Therefore, by Proposition 3.2.1, we conclude that $x \in \mathrm{Pic}(S)$. □

Symplectic automorphisms of K3 surfaces were first introduced by Nikulin [Nik79]. He analyzed the properties of symplectic automorphisms of finite order through their induced action on the second cohomology group and proved that this action depends uniquely on the order of the automorphism, as shown in [Nik79, Theorem 4.7]. Additionally, he explored the existence and uniqueness of these actions on cohomology from a lattice-theoretical perspective, proving that the induced isometry on $H^2(S,\mathbb{Z})$ is unique up to isometry. As a consequence of his work, we have the following result.

Proposition 3.3.9. If a K3 surface S admits a non-trivial symplectic automorphism $f \in \mathrm{Aut}(S)$ of finite order, then $\rho(S) \geq 9$.

The case of non-symplectic automorphisms has been treated in various works, such as [Nik83, OZ98, MO98, AS08, OZ11, Tak11, AST11, GS13]. In particular, non-symplectic automorphisms of prime order on K3 surfaces are classified in [AST11], where the invariant lattice and the topological structure of the fixed locus are described.

We define the subgroup $\mathrm{Aut}^{\pm}(S) \subset \mathrm{Aut}(S)$ consisting of all symplectic and anti-symplectic automorphisms of S, i.e., for any $f \in \mathrm{Aut}^{\pm}(S)$, either $f^*\omega_S = \omega_S$ or $f^*\omega_S = -\omega_S$. The main properties of $\mathrm{Aut}^{\pm}(S)$ are summarized in the following lemma.

Lemma 3.3.10. Let S be a K3 surface. Then:

1. $\mathrm{Aut}^{\pm}(S)$ is a finite index subgroup of $\mathrm{Aut}(S)$.
2. There is a one-to-one correspondence

$$\mathrm{Aut}^{\pm}(S) \cong \{(\varphi_P, \varepsilon) \in O(\mathrm{Pic}(S)) \times \mu_2 | \varphi_P \text{ preserves } \mathrm{Amp}(S) \text{ and } \overline{\varphi}_P = \varepsilon\}.$$

   Here μ_2 denotes the multiplicative cyclic group of order 2.

Proof. To prove that $\mathrm{Aut}^{\pm}(S)$ is a finite-index subgroup of $\mathrm{Aut}(S)$ we refer to [Nik83, Theorem 10.1.2]. Furthermore, by combining Remarks 2.1.12, 3.3.4 and Corollary 3.3.7 we can describe $\mathrm{Aut}^{\pm}(S)$ in terms of isometries of $\mathrm{Pic}(S)$. □

3.4 Lattice polarized K3 surfaces

All K3 surfaces are topologically the same. Indeed, as observed in Section 3.2, all topological invariants such as Betti numbers, the intersection forms, and Hodge numbers are independent of the specific K3 surface. Moreover, Kodaira proved in [Kod64, Theorem 13] that any two K3 surfaces are deformation equivalent, which, by Ehresmann's theorem, implies that all K3 surfaces are diffeomorphic. Thanks to the Weak Torelli theorem

(Proposition 3.4.4 below), we know that K3 surfaces are distinguished by their Hodge decomposition, which in turn determines their complex structure.

In this section, we briefly describe the period domain and the moduli space of marked K3 surfaces, as well as the moduli space of lattice polarized K3 surfaces, which parametrize families of K3 surfaces. For more details, we refer to [Dol96], [DK07], and [Huy16].

Given a K3 surface S, the isomorphism $H^2(S,\mathbb{Z}) \cong \Lambda_{K3}$ is neither unique nor canonical. A chosen such isomorphism is referred to as a *marking*.

Definition 3.4.1 (Marked K3 surfaces)**.** A *marked K3 surface* is a pair (S,ϕ) where S is a K3 surface and $\phi\colon H^2(S,\mathbb{Z}) \to \Lambda_{K3}$ an isometry.

Now, consider the one-dimensional space $H^{2,0}(S) = \mathbb{C}\omega_S$, where ω_S is the unique (up to scalar multiplication) nowhere vanishing global rational 2-form. This space is called as the *period line* of S. It satisfies the following conditions, known as the Riemann relations of the period:

$$\langle \omega_S, \omega_S \rangle = 0 \quad \text{and} \quad \langle \omega_S, \overline{\omega}_S \rangle > 0. \tag{3.1}$$

Let (S,ϕ) be a marked K3 surface. The $\mathbb{C}$-linear extension of ϕ to $H^2(S,\mathbb{C})$ gives an isomorphism $\phi_{\mathbb{C}}\colon H^2(S,\mathbb{C}) \to \Lambda_{K_3} \otimes \mathbb{C}$. As usual, we set $\Lambda_{\mathbb{C}} = \Lambda_{K_3} \otimes \mathbb{C}$. Consequently, the image of the period line $\mathbb{C}\omega_S$ corresponds to an element of $\Lambda_{\mathbb{C}}$ which also satisfies (3.1). Two marked surfaces (S,ϕ) and (S',ϕ') are said to be *equivalent* if there exists an isomorphism $f\colon S \to S'$ such that $\phi' = \phi \circ f^*$. It is straightforward to verify that the respective images of the period lines in $\Lambda_{\mathbb{C}}$ coincide.

We now define the period domain. For any element $\omega \in \Lambda_{\mathbb{C}}$, we denote by $[\omega] \in \mathbb{P}(\Lambda_{\mathbb{C}})$ its corresponding point.

Definition 3.4.2 (Period domain)**.** The *period domain* is the set

$$\mathcal{D} = \{[\omega] \in \mathbb{P}(\Lambda_{\mathbb{C}}) | \langle \omega, \omega \rangle = 0,\ \langle \omega, \overline{\omega} \rangle > 0\} \subset \mathbb{P}(\Lambda_{\mathbb{C}}) \cong \mathbb{P}^{21}.$$

The period domain $\mathcal{D}$ is an open set (in the usual topology) of the smooth quadric hypersurface $\mathcal{Q} \subset \mathbb{P}^{21}$ defined by the equation $z_0^2 + z_1^2 + z_2^2 - z_3^3 - \cdots - z_{21}^2 = 0$, since Λ_{K3} has signature $(3,19)$. Thus, $\mathcal{D}$ is a 20-dimensional complex manifold and can be interpreted as the space of Hodge structures of Λ_{K3}, which corresponds to the space of Hodge structures of K3 surfaces. Let $\mathcal{M}$ be denote the set of marked K3 surfaces. Given marked K3 surface (S,ϕ), the period line $\mathbb{C}\omega_S$ corresponds to a point $\phi_{\mathbb{C}}(\mathbb{C}\omega_S) \in \mathcal{D}$. This association defines a natural map $p\colon \mathcal{M} \to \mathcal{D}$, known as the *period map*.

Theorem 3.4.3 (Surjectivity of period map)**.** The period map $p\colon \mathcal{M} \to \mathcal{D}$ is surjective.

In other words, every point in the period domain $\mathcal{D}$ occurs as the period line of a marked K3 surface. Note that the position of the period line of a K3 surface in the period domain depends on the marking. Indeed, given a marked K3 surface (S,ϕ), consider any isometry $\sigma \in O(\Lambda_{K3})$. Then, $\sigma \circ \phi$ defines a new marking of S, and the marked K3 surface $(S, \sigma \circ \phi)$ has a period line that corresponds to a different point in $\mathcal{D}$. On the other hand, we have a weaker version of the Global Torelli Theorem.

Proposition 3.4.4 (Weak Torelli Theorem)**.** Let S and S' be two K3 surfaces and $\varphi\colon H^2(S',\mathbb{Z}) \to H^2(S,\mathbb{Z})$ be an isometry. Assume that φ is a Hodge isometry, i.e., the period line $\mathbb{C}\omega_{S'}$ is sent to the period line $\mathbb{C}\omega_S$ under the $\mathbb{C}$-linear extension $\varphi_{\mathbb{C}}$. Then S and S' are isomorphic.

Hence, the Weak Torelli theorem asserts that two K3 surfaces are isomorphic if and only if there exist markings for them such that they are equivalent marked surfaces, meaning their corresponding period points coincide. Combining this with Theorem 3.4.3, we aim to construct a moduli space of K3 surfaces where identified period points correspond to isomorphic K3 surfaces. The natural approach is to quotient the space of marked K3 surfaces, $\mathcal{M}$, by the equivalence relation, and to quotient the period domain, $\mathcal{D}$, by the action of the group $O(\Lambda_{K3})$. In this way, the period map naturally descends to the quotient spaces. However, the quotient $\mathcal{D}/O(\Lambda_{K3})$ fails to be Hausdorff because the action of $O(\Lambda_{K3})$ on $\mathcal{D}$ is not properly discontinuous [Huy16, Chapter 6, 1.3.]. To address this issue, we will focus on a specific subclass of K3 surfaces with more favorable properties: the lattice polarized K3 surfaces. This notion was introduced by Dolgachev in [Dol96].

Let $L \hookrightarrow \Lambda_{K3}$ be a primitive even non-degenerate sublattice of rank ρ and signature $(1, \rho-1)$. Note that L is a hyperbolic lattice. Since L has signature $(1, \rho-1)$, there is a basis of $L_{\mathbb{R}}$ where the extension of the bilinear form is diagonalizable with entries $(1, -1, \ldots, -1)$. Thus, the cone $\mathcal{C}(L) := \{x \in L_{\mathbb{R}} | x^2 > 0\}$ consists of two disjoint connected components, $\mathcal{C}^+(L)$ and $\mathcal{C}^-(L)$, so that $\mathcal{C}(L) = \mathcal{C}^+(L) \sqcup \mathcal{C}^-(L)$. Two elements $x, y \in \mathcal{C}(L)$ belong to the same component if and only if $x \cdot y > 0$. We define the *set of roots* $\Delta(L) := \{\delta \in L | \delta^2 = -2\}$, which can also be written as a disjoint union $\Delta(L) = \Delta^+(L) \sqcup \Delta^-(L)$, where $\Delta^+(L)$ is the component for which $x \cdot \delta \geq 0$ for all $x \in \mathcal{C}^+(L)$ and $\delta \in \Delta^+(L)$. We define $\mathcal{A}(L) := \{x \in \mathcal{C}^+(L) \cap L | x \cdot \delta > 0, \text{ for all } \delta \in \Delta^+(L)\}$.

When $L \cong \mathrm{Pic}(S)$ for a K3 surface S, $\Delta^+(L)$ corresponds to the set of effective divisors with self-intersection -2. In particular, it contains all the rational curves on S. Moreover, $\mathcal{A}(L)$ corresponds to the ample cone $\mathrm{Amp}(S)$, as stated in Proposition 3.1.6.

Definition 3.4.5 (Lattice polarized K3 surfaces)**.** An L-polarized K3 surface is a pair (S, j) consisting of a K3 surface S and a primitive embedding $j\colon L \hookrightarrow \mathrm{Pic}(S)$ such that $j(\mathcal{C}^+(L)) \cap \overline{\mathrm{Amp}(S)} \neq \emptyset$. When $j(\mathcal{C}^+(L)) \cap \mathrm{Amp}(S) \neq \emptyset$ we say that (S, j) is an *ample L-polarized K3 surface*.

We say that two M-polarized K3 surfaces (S, j) and (S', j') are *equivalent* if there exists an isomorphism $f\colon S \to S'$ such that $j = f^* \circ j'$. In order to construct the desired moduli space of L-polarized K3 surfaces, we introduce the following definition.

Definition 3.4.6 (Marked lattice polarized K3 surfaces)**.** A *marked L-polarized K3 surface* is a marked surface (S, ϕ) such that $\phi^{-1}(L) \subset \mathrm{Pic}(S)$ and (S, j_ϕ) is a M-polarized K3 surface, where $j_\phi := \phi^{-1}|_{\phi^{-1}(L)}\colon \phi^{-1}(L) \hookrightarrow \mathrm{Pic}(S)$.

We say that two marked L-polarized surfaces (S, ϕ) and $((S', \phi')$ are *equivalent* if there exists an isomorphism $f\colon S \to S'$ such that $\phi' = \phi \circ f^*$, and so, $j_\phi = f^* \circ j_{\phi'}$.

Now we define the corresponding period domain. Let $N = L^\perp$ be the orthogonal complement of L inside Λ_{K3}. The period domain for L-polarized K3 surfaces is defined by the set

$$\mathcal{D}_L := \{[\omega] \in \mathbb{P}(N_{\mathbb{C}}) | \langle \omega, \omega \rangle = 0,\ \langle \omega, \overline{\omega} \rangle > 0\} \subset \mathcal{D},$$

which is an open subset (in the usual topology) of a smooth quadric hyperplane in $\mathbb{P}^{21-\rho}$. Thus, $\mathcal{D}_L$ is a complex maninfold of dimension $20 - \rho$. Note that for a marked L-polarized surface (S, ϕ), $\phi_{\mathbb{C}}(\mathbb{C}\omega_S) \in \mathcal{D}_L$, since after identifying L as a primitive sublattice of $\mathrm{Pic}(S)$, we have $T(S) \subset N$ and $H^{2,0}(S) = \mathbb{C}\omega_S \subset N_{\mathbb{C}}$.

Let $\mathcal{M}_L \subset \mathcal{M}$ be the set of marked L-polarized K3 surfaces. Thus, we have a period map $p_L \colon \mathcal{M}_L \to \mathcal{D}_L$ which is the restriction of the period map p defined above.

Proposition 3.4.7. Every point of $\mathcal{D}_L$ is realized as the period line of some marked L-polarized K3 surface.

Again, the image of the period line of an L-polarized under the period map depends on the marking ϕ. Indeed, given an L-polarized K3 surface (S, j) and an isometry $\sigma \in O(\Lambda_{K3})$ acting trivially on L, the isometry $\sigma \circ \phi$ is another marking of (S, j). Let $\Gamma(L) := \{\sigma \in O(\Lambda_{K3}) | \sigma(l) = l, \text{ for all } l \in L\}$. The restriction of elements in $\Gamma(L)$ to N defines a natural injective homomorphism $\Gamma(L) \to O(N)$. Let Γ_L be the image of this map. By Proposition 2.1.11, Γ_L coincides with the subgroup of isometries of N acting trivially on the discriminant group $A(N) \cong A(L)$.

The group Γ_L is a discrete group which acts properly discontinuously on $\mathcal{D}_L$ and so the quotient $\mathcal{D}_L/\Gamma_L$ is nicely defined. It is indeed a $(20 - \rho)$-dimensional quasi-projective variety. We consider then the quotient $\mathcal{M}_L/\Gamma(L)$, which corresponds to the set of equivalence classes of L-polarized K3 surfaces. Thus, the period map descends to a bijection

$$\mathcal{M}_L/\Gamma(L) \cong \mathcal{D}_L/\Gamma_L.$$

Therefore, $\mathcal{M}(L) := \mathcal{M}_L/\Gamma(L)$ is the *coarse moduli space of L-polarized K3 surfaces* (we refer to [Dol96, Section 3] for more details on this construction).

By [Nik79, heorem 1.14.4], any even non-degenerate lattice L of rank $\rho < 11$ and signature $(1, \rho - 1)$ in primitively embedded in Λ_{K3}. Hence, as a consequence of the theory outlined above, we find that for any such lattice there exists a $(20 - \rho)$-dimensional family of K3 surfaces in which L is primitively embedded in the Picard lattice. The following result confirms that L can indeed occur as the Picard lattice of a K3 surface.

Theorem 3.4.8 ([Mor84b, Corollary 2.9])**.** Let L be an even lattice of signature $(1, \rho - 1)$, with $\rho \leq 10$. Then there exists a K3 surface S and a lattice isometry $\mathrm{Pic}(S) \cong L$.

Indeed, the generic member S of that $(20 - \rho)$-dimensional family satisfies $\mathrm{Pic}(S) \cong L$. Now, we introduce the concept of Aut-general surfaces.

Definition 3.4.9. Let S be a K3 surface, and write $H^{2,0}(S) = \mathbb{C} \cdot \omega_S$. We say that S is *Aut-general* if, for every $f \in \mathrm{Aut}(S)$, one has $f^*\omega_S = \pm\omega_S$.

In other words, a K3 surface S is said to be Aut-general if every automorphism of S is either symplectic or anti-symplectic, i.e., $\mathrm{Aut}(S) = \mathrm{Aut}^{\pm}(S)$. One advantage of considering Aut-general K3 surfaces is that their automorphism group can be determined from the isometries of the Picard lattice (see Lemma 3.3.10(2)), rather than from the isometries of the second cohomology group, which is a larger lattice.

Proposition 3.4.10. Every K3 surface S with odd Picard rank is Aut-general.

Proof. The proof presented here is based on the proof in [Huy16, Chapter 3, Corollary 3.5]. Let S be a K3 surface with odd Picard rank and let $f \in \mathrm{Aut}(S)$ be a non-trivial automorphism. Assume, by contradiction, that $f^*\omega_S = \lambda\omega_S$, where $\lambda \neq \pm 1$.

We observe that for any eigenvalue γ of the action of $f^*_{\mathbb{C}}$ on $\mathrm{T}(S)_{\mathbb{C}}$, its complex conjugate $\overline{\gamma}$ is also an eigenvalue, since the characteristic polynomial associated with $f^*_{\mathbb{C}}$ has integer coefficients. Consequently, the number of eigenvalues $\gamma \neq \pm 1$, counted with multiplicity, must be even. Since $\mathrm{T}(S)$ has odd rank, $\gamma = 1$ or $\gamma = -1$ must occur as an eigenvalue. Let $0 \neq x \in \mathrm{T}(S)$ be an eigenvector associated with this integer eigenvalue γ. From $\langle x, \omega_S\rangle = \langle f^*x, f^*\omega_S\rangle = \pm\lambda\langle x, \omega_S\rangle$, it follows that $\langle x, \omega_S\rangle = 0$. This implies $x \in \mathrm{Pic}(S)$, which contradicts the fact that $\mathrm{Pic}(S) \cap \mathrm{T}(S) = \{0\}$. □

The previous result may not hold for K3 surfaces with even Picard rank, as we will see in the following example.

Example 3.4.11. Let $\zeta \neq 1$ be a primitive 3rd root of unity and consider the hyperbolic lattice $L = U(3) \oplus A_2$ of rank 4, where A_2 is the lattice with intersection matrix

$$\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}.$$

In [AS08, Theorem 3.3, Proposition 4.9], it is established the existence of a smooth quartic surface $S \subset \mathbb{P}^3$ with a non-symplectic automorphism f of order 3, such that $f^*\omega_S = \zeta\omega_S$. This non Aut-general K3 surface is defined by the equation $F_4(x_0, x_1, x_2) + F_1(x_0, x_1x_2)x_3^3 = 0$, where F_4 and F_1 are general homogeneous polynomials in $\mathbb{C}[x_0, x_1, x_2]$ of degree 4 and 1, respectively. The automorphism f is given by $f(x_0, x_1, x_2, x_3) = (x_0, x_1, x_2, \zeta x_3)$.

The condition of a K3 surface S being Aut-general is a very general condition. By *very general* we mean that the period line lies outside of a specific enumerable union of hypersurfaces inside the period domain. Therefore, the existence of an automorphism $f \in \mathrm{Aut}(S)$ with $f^*\omega_S \neq \pm\omega_S$ imposes an algebraic constraint on the position of the period line in the period domain. Next, we give an idea of this fact.

Fix a hyperbolic lattice L and an L-polarized K3 surface S such that $\mathrm{Pic}(S) = L$. Let $f \in \mathrm{Aut}(S) \setminus \mathrm{Aut}^{\pm}(S)$ be an automorphism of S satisfying $f^*\omega_S = \lambda\omega_S$, where $\lambda \neq \pm 1$. It is known that $\lambda^n = 1$ for some integer $n \geq 3$. Observe that λ is an eigenvalue of the action of f on the complex space $T(S)_{\mathbb{C}} = N_{\mathbb{C}}$. In other words, λ is a root of the characteristic polynomial of $f^*_{\mathbb{C}}|_{N_{\mathbb{C}}}$, which has integer coefficients. Consequently, the complex conjugate $\overline{\lambda}$ is also a root of this polynomial and thus an eigenvalue of $f^*_{\mathbb{C}}|_{N_{\mathbb{C}}}$. This implies that the eigenspace V_λ, associated to λ, is a proper subspace of $T(S)_{\mathbb{C}} = N_{\mathbb{C}}$. Furthermore, since ω_S is an eigenvector corresponding to λ, we have $\mathbb{C}\omega_S \subset V_\lambda \subsetneq N_{\mathbb{C}}$. As a result, the period line $\mathbb{C}\omega_S$ must lie inside a closed subset of $\mathcal{D}_L/\Gamma_L$.

3.5 K3 surfaces with Picard rank two

In this section, we focus on K3 surfaces with Picard rank two. We provide a more detailed description of the automorphisms and the Picard lattice of a K3 surface that admits non-symplectic automorphisms of prime order. Specifically, we characterize automorphisms of order 2.

We begin by describing the Mori cone $\overline{\mathrm{NE}}(S)$ of a K3 surface S with Picard rank two. This cone is generated by two elements.

Proposition 3.5.1 ([Kov94, Theorem 2])**.** Let S be a K3 surface with $\rho(S) = 2$. Then the Mori cone $\overline{\mathrm{NE}}(S)$ is generated by either

1. two classes of rational curves, or
2. a class of a rational curve and a class of an elliptic curve, or
3. two classes of elliptic curves, or
4. two non effective classes in $\mathrm{Pic}(S)_{\mathbb{R}}$ if S does not contains neither rational nor elliptic curves.

Let S be a K3 surface with Picard rank $\rho(S) = 2$ and let H be a primitive ample class in $\mathrm{Pic}(S)$. Such ample class exists. Indeed, since S is projective, let H be a very ample class such that $\varphi_{|H|}\colon S \hookrightarrow \mathbb{P}^N$. If H is not primitive, it is a multiple of a primitive divisor class H', which is therefore ample. Hence, without loss of generality, we assume H to be primitive. By Lemma 2.1.17, the ample class extends to a basis $\{H, W\}$ of $\mathrm{Pic}(S)$. In this basis, the intersection product is given by

$$Q = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}, \tag{3.2}$$

for some integers $a, b, c \in \mathbb{Z}$. For convenience, we refer to the opposite of the discriminant of $\mathrm{Pic}(S)$ as the *discriminant* of S and denote it by $r(S) = -\operatorname{disc}(\mathrm{Pic}(S))$. This is a positive integer since $\mathrm{Pic}(S)$ has signature $(1, 1)$.

Proposition 3.5.2. Let S be a K3 surface with $\rho(S) = 2$. Let D be an effective divisor with $D^2 \geq 0$ and such that $\mathrm{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}D$, where H is a very ample divisor with $4 \leq H^2 \leq 34$. Then, either D is nef, $|D|$ is base point free and a general element of $|D|$ is smooth; or $D = H + \Gamma$ is not nef, where Γ is a rational curve with $H \cdot \Gamma = 1$, in this case $r(S) = 2H^2 + 1$.

Proof. Write $D = M + F$, where M and F are the mobile and fixed parts respectively. The mobile part M is an effective nef divisor such that $M^2 \geq 0$ by Proposition 3.1.8(1). Assume $F \neq 0$. combining Proposition 3.1.8(2) and Proposition 3.5.1 we deduce that $F = a\Gamma_1 + b\Gamma_2$ is a positive sum of at most two rational curves since S has $\rho(S) = 2$. We consider two cases: when M is nef and big ($M^2 > 0$) or $M^2 = 0$.

Assume we are in the latter case. Thus, $M = mE$, where E is a curve with $p_a(E) = 1$, and $m \geq 1$ by 3.1.8(4). This implies that we are in the case (2) of Proposition 3.5.1, i.e., the Mori cone $\overline{\mathrm{NE}}(S)$ is generated by only one rational curve, say $\Gamma = \Gamma_1$, and E. Thus $F = a\Gamma$ and $\Gamma \cdot E = 1$, by [SD74, (2.7.4)]. Since $H \in \overline{\mathrm{NE}}(S)$, we have $H = \alpha E + \beta\Gamma$ with $\alpha, \beta > 0$. Notice that $\beta = H \cdot E \geq 3$; otherwise, it contradicts Proposition 3.1.7. Moreover

$$0 < H \cdot \Gamma = \alpha - 2\beta, \qquad \text{and so} \qquad H^2 = 2\beta(\alpha - \beta) \geq 4\beta^2 \geq 36.$$

This leads to a contradiction. Therefore, M is nef and big, and so $D^2 \leq M^2$ by 3.1.8(3). Now, we consider the two cases where M is a multiple of H or not.

If M is not proportional to H, we find that $\langle H, M\rangle$ is a rank 2 hyperbolic sublattice of $\mathrm{Pic}(S)$. Then

$$k((H\cdot D)^2 - H^2D^2) = -k\,\mathrm{disc}(H,D) \overset{(*)}{=} -\mathrm{disc}(H,M) = (H\cdot M)^2 - H^2M^2 \leq (H\cdot D)^2 - H^2D^2,$$

where the equality $(*)$ follows from Proposition 2.1.1(3). Hence, the previous inequality can only be satisfied when $k=1$ and $H\cdot D = H\cdot M$, i.e., when $D = M$ has no fixed part. Thus, D is nef, $|D|$ is base point free, by 3.1.8(1). Consequently, a general member is a smooth curve.

If $M = \lambda H$, $\mathrm{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}F$. From $(\lambda H + F)^2 = D^2 \leq M^2 = \lambda^2 H^2$, it follows that $F^2 \leq -2\lambda H\cdot F < 0$. Let $\Gamma = \Gamma_i$ be one of the rational curves in the support of F, thus $F^2 \leq \Gamma^2 = -2$ and $H\cdot\Gamma \leq H\cdot F$. Thus we have the following for some positive integer l

$$l((H\cdot F)^2 - H^2F^2) = -l\,\mathrm{disc}(H,F) = -\mathrm{disc}(H,\Gamma) = (H\cdot\Gamma)^2 - H^2\Gamma^2 \leq (H\cdot F)^2 - H^2F^2.$$

Then $l=1$, $H\cdot F = H\cdot\Gamma$ and so $F=\Gamma$. Therefore, $\lambda = 1 = H\cdot\Gamma$, $D = H+\Gamma$, D is not nef, and since $\mathrm{Pic}(S)$ is also generated by H and Γ we get $r(S) = \mathrm{disc}(H,\Gamma) = 2H^2+1$. □

3.5.1 Automorphisms

As mentioned above, for every automorphism $f \in \mathrm{Aut}(S)$, the isometry f^* preserves the Hodge structure of $\mathrm{T}(S)$. By [Huy16, Chapter 3, Corollary 3.4, and Chapter 15, Corollary 1.14], the subgroup of $O(\mathrm{T}(S))$ consisting of all Hodge isometries of $\mathrm{T}(S)$ is the finite cyclic group μ_m, where $\phi(m)$ divides $\mathrm{rank}(\mathrm{T}(S))$. Here, ϕ is the Euler's totient function. Thus, the Hodge isometry f^* has finite order n on $\mathrm{T}(S)$, with n dividing m. Let p be a prime factor of m. The possible values for m and p when $\mathrm{rank}(T(S)) = 20$ are shown in the following table.

m	2	3	4	5	6	8	10	11	12	25	33	44	50	66
$\phi(m)$	1	2	2	4	2	4	4	10	4	20	20	20	20	20
p	2	3	2	5	2, 3	2	2, 5	11	2, 3	5	3, 11	2, 11	2, 5	2, 3, 11

In particular, if f is an automorphism of prime order p, we conclude that $p \in \{2,3,5,11\}$. This is summarized in the following proposition, where we also describe the respective invariant lattices.

Proposition 3.5.3. Let S be a K3 surface with $\rho(S) = 2$ and f be an automorphism of finite prime order p. Then f is non-symplectic, $H^2(S,\mathbb{Z})^{f^*} \subseteq \mathrm{Pic}(S)$, and $p \in \{2,3,5,11\}$. Moreover,

1. If $p=2$, then $H^2(S,\mathbb{Z})^{f^*}$ is isometric to $\langle 2\rangle$, U, $U(2)$ or $\langle 2\rangle \oplus \langle -2\rangle$.
2. If $p=3$, then $H^2(S,\mathbb{Z})^{f^*}$ is isometric to U or $U(3)$.
3. If $p=5$, then $H^2(S,\mathbb{Z})^{f^*}$ is isometric to H_5.
4. If $p=11$, then $H^2(S,\mathbb{Z})^{f^*}$ is isometric to U or $U(11)$.

The invariant lattice $H^2(S,\mathbb{Z})^{f^*} = \mathrm{Pic}(S)$ as long as it has rank two, i.e., in all cases above except when $H^2(S,\mathbb{Z})^{f^*} = \langle 2\rangle$.

Proof. Let $f \in \mathrm{Aut}(S)$ be an automorphism of prime order p. By Proposition 3.3.9, it is non-symplectic. Moreover, from Corollary 3.3.8, the invariant and co-invariant lattices are non-trivial and satisfy

$$H^2(S,\mathbb{Z})^{f^*} \subset \mathrm{Pic}(S) \quad \text{and} \quad \mathrm{T}(S) \subset H^2(S,\mathbb{Z})_{f^*}.$$

The action of f^* on the transcendental lattice $\mathrm{T}(S)$ induces a non-trivial Hodge isometry of order p. We conclude then that $p \in \{2,3,5,11\}$. Recall from Corollary 3.3.8(3) that $H^2(S,\mathbb{Z})^{f^*}$ is a hyperbolic p-elementary lattice of rank one or two. If $p = 2$, (1) follows from Lemma 2.1. When $p \neq 2$, $H^2(S,\mathbb{Z})^{f^*}$ has rank two; otherwise, $H^2(S,\mathbb{Z})^{f^*} = \langle D \rangle$, for a divisor class $D^2 = p$, which contradicts the fact that the intersection product of $\mathrm{Pic}(S)$ is even. Finally, assertions (2), (3) and (4) follow from the classification of non-symplectic automorphisms of order p given by Artebani, Sarti and Taki; see [AS08, Table 2] and [AST11, Tables 2 and 4] □

Using the characterization of $\mathrm{Aut}^{\pm}(S)$ in Lemma 3.3.10, along with the classical theory of binary quadratic forms, Galluzzi, Lombardo and Peters proved that when S is a K3 surface with Picard rank two, the subgroup $\mathrm{Aut}^{\pm}(S) \subset \mathrm{Aut}(S)$ has four possible structures.

Proposition 3.5.4 ([GLP10, Corollary 1])**.** Let S be a K3 surface with $\rho(S) = 2$. Then,

$$\mathrm{Aut}^{\pm}(S) \cong \begin{cases} \{1\} \text{ or } \mathbb{Z}_2 & \text{if and only if} \quad \exists D \in \mathrm{Pic}(S) \text{ with } D^2 \in \{0,-2\} \\ \mathbb{Z} \text{ or } \mathbb{Z}_2 * \mathbb{Z}_2 & \text{if and only if} \quad \nexists D \in \mathrm{Pic}(S) \text{ with } D^2 \in \{0,-2\} \end{cases}$$

Automorphisms of order two belongs to $\mathrm{Aut}^{\pm}(S)$. Involutions on K3 surfaces have been classified by Nikulin in [Nik79, Nik83] from a lattice theoretical viewpoint. The following result summarizes this classification for the specific case of Picard rank two. For K3 surfaces with Picard rank two, where the Picard lattice is not 2-elementary, this provides a necessary and sufficient condition for the existence of involutions.

Proposition 3.5.5. Let S be a smooth K3 surface with Picard rank $\rho(S) = 2$. Then

1. The Picard lattice $\mathrm{Pic}(S)$ is isomorphic to U, $U(2)$ or $\langle 2 \rangle \oplus \langle -2 \rangle$ if and only if there exists a non-trivial involution $f \in \mathrm{Aut}(S)$ whose invariant lattice $H^2(S,\mathbb{Z})^{f^*}$ is $\mathrm{Pic}(S)$. In particular, this is the unique non-trivial involution of S.
2. Assume that $\mathrm{Pic}(S)$ is neither isomorphic to U, $U(2)$ nor $\langle 2 \rangle \oplus \langle -2 \rangle$, and let $f \in \mathrm{Aut}(S)$ be a non-trivial involution. Then, $H^2(S,\mathbb{Z})^{f^*} = \langle A \rangle$ for some ample divisor $A \in \mathrm{Pic}(S)$ with $A^2 = 2$. More precisely, there is a bijection between involutions f of S and ample divisors A with $A^2 = 2$.

Proof. We start by observing that f is a non-symplectic involution with invariant lattice $H^2(S,\mathbb{Z})^{f^*} \subseteq \mathrm{Pic}(S)$ isometric to either $\langle 2 \rangle$, U, $U(2)$ or $\langle 2 \rangle \oplus \langle -2 \rangle$, by Proposition 3.5.3(1).

Now we prove (1). First, we note that $\mathrm{Pic}(S)$ is a 2-elementary lattice exactly when it is one of the three possibilities: U, $U(2)$ or $\langle 2 \rangle \oplus \langle -2 \rangle$. This follows from the Nikulin classification of 2-elementary lattices with signature $(1,1)$ (see Lemma 2.1). Assume this is the case. So, id and $-$id are the same as automorphisms of the discriminant group $A(\mathrm{Pic}(S))$ since it is isomorphic to $(\mathbb{Z}_2)^l$. Thus, by Proposition 2.1.11 and Theorem 3.3.3, there exists an automorphism $f \in \mathrm{Aut}(S)$ whose induced isometry f^* of $H^2(S,\mathbb{Z})$ acts on $\mathrm{Pic}(S)$ as id, and on $\mathrm{T}(S)$ as $-$id. Clearly, f is a non-trivial involution with $H^2(S,\mathbb{Z})^{f^*} = \mathrm{Pic}(S)$. The converse follows from the fact that the existence of a non-trivial involution f with $H^2(S,\mathbb{Z})^{f^*} = \mathrm{Pic}(S)$ implies that $\mathrm{Pic}(S)$ is a 2-elementary lattice. Finally, for each of these three lattices, one can find a divisor with self-intersection 0 or -2. So, $\mathrm{Aut}^{\pm}(S) = \mathbb{Z}_2$, which implies the uniqueness of the non-trivial involution above.

Assume the hypothesis in (2) now. Since $H^2(S,\mathbb{Z})^{f^*}$ is a non-trivial primitive sublattice of $\mathrm{Pic}(S)$, either it has rank one or it is $\mathrm{Pic}(S)$. It is not the latter, otherwise (1) holds contradicting the hypothesis. Then, $H^2(S,\mathbb{Z})^{f^*} = \langle 2\rangle$. That is, there exists a divisor class $A \in \mathrm{Pic}(S)$ with $A^2 = 2$ such that $\mathrm{Pic}(S) = \langle A\rangle$. This divisor class A is ample since for any ample class D we have that $D + f^*D$ is a multiple of A. This proves the first assertion of (2) and one direction of the bijection. For the other direction, consider an ample divisor $A \in \mathrm{Pic}(S)$ with $A^2 = 2$. Since S is a smooth K3 surface, results of Saint-Donat [SD74, Proposition 2.6, Theorems 3.1 and 5.1] establish that the rational map corresponding to the complete linear system $|A|$ identifies S as a double cover of $\mathbb{P}^2$ branched along a smooth sextic curve (see Example 3.1.3), and it induces naturally an involution f. Moreover, it is easy to see that $f^*A = A$. Finally, to see that this is a 1-1 correspondence, it is enough to observe that the action of an involution f on $\mathrm{Pic}(S)$ is the reflection along the line generated by its corresponding ample divisor A: $E \longmapsto (A\cdot E)A - E$, which uniquely determines f. □

Proposition 3.5.4 implies that the finiteness of $\mathrm{Aut}^{\pm}(S)$, and then of $\mathrm{Aut}(S)$, is equivalent to the existence of a divisor class $D \in \mathrm{Pic}(S)$ with $D^2 \in \{0,-2\}$, putting this together with Proposition 3.5.5 we can determine completely $\mathrm{Aut}^{\pm}(S)$.

Corollary 3.5.6. Let S be a K3 surface with $\rho(S) = 2$. Then, if $\mathrm{Pic}(S) = U, U(2), \langle 2\rangle \oplus \langle -2\rangle$ the subgroup $\mathrm{Aut}^{\pm}(S) \subset \mathrm{Aut}(S)$ is isomorphic to $\mathbb{Z}_2$, otherwise, $\mathrm{Aut}^{\pm}(S)$ is isomorphic to

$$\left.\begin{array}{l} \{1\} \\ \mathbb{Z}_2 \\ \mathbb{Z} \\ \mathbb{Z}_2 * \mathbb{Z}_2 \end{array}\right\} \text{ if and only if } \left\{\begin{array}{l} \exists D \in \mathrm{Pic}(S) \text{ with } D^2 \in \{-2,0\} \text{ and } \nexists A \in \mathrm{Pic}(S) \text{ ample with } A^2 = 2, \\ \exists D \in \mathrm{Pic}(S) \text{ with } D^2 \in \{-2,0\} \text{ and } \exists A \in \mathrm{Pic}(S) \text{ ample with } A^2 = 2, \\ \nexists D \in \mathrm{Pic}(S) \text{ with } D^2 \in \{-2,0\} \text{ and } \nexists A \in \mathrm{Pic}(S) \text{ ample with } A^2 = 2, \\ \nexists D \in \mathrm{Pic}(S) \text{ with } D^2 \in \{-2,0\} \text{ and } \exists A \in \mathrm{Pic}(S) \text{ ample with } A^2 = 2. \end{array}\right.$$

Let us follow the notation in (3.2). That is, we assume that the $\mathrm{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}W$, where H is an ample class, and the intersection matrix in given by the matrix Q in (3.2). Without loss of generality, we assume that $c \neq 0$. In the next proposition, we describe every element in $\mathrm{Aut}^{\pm}(S)$ via its action on $\mathrm{Pic}(S)$ in this basis.

Proposition 3.5.7 ([Lee23, Theorem 1.1, Theorem 1.2 and Lemma 2.7])**.** Let S be a K3 surface with $\rho(S) = 2$, $\mathrm{Pic}(S) \neq U, U(2), \langle 2\rangle \oplus \langle -2\rangle$. Let H and Q be as above.

An isometry $\phi \in O(Pic(S))$ is induced by an automorphism $f \in \mathrm{Aut}^{\pm}(S)$ if and only if

$$(\phi + \mathrm{id}) * Q^{-1} \in M_{2\times 2}(\mathbb{Z}) \text{ or } (\phi - \mathrm{id}) * Q^{-1} \in M_{2\times 2}(\mathbb{Z}), \text{ and } \phi(H) \text{ is ample.}$$

(These are exactly the Gluing and Torelli conditions.)

Furthermore, we have the following characterizations of involutions and automorphisms of infinite order.

1. The automorphism $f \in \mathrm{Aut}^{\pm}(S)$ is an involution if and only if the corresponding isometry $\phi = f^*$ is of

the form

$$\phi = \begin{pmatrix} \alpha & \beta \\ -\frac{b}{c}\alpha + \frac{a}{c}\beta & -\alpha \end{pmatrix},$$

where (α, β) is an integer solution of the equation:

$$\alpha^2 - \frac{b}{c}\alpha\beta + \frac{a}{c}\beta^2 = 1. \tag{$*$}$$

2. The automorphism $f \in \mathrm{Aut}^{\pm}(S)$ has infinite order if and only if the corresponding isometry $\phi = f^*$ is of the form

$$\phi = \begin{pmatrix} \alpha & \beta \\ -\frac{a}{c}\beta & \alpha - \frac{b}{c}\beta \end{pmatrix},$$

where (α, β) is an integer solution of the equation $(*)$. In this case, ϕ is a power of

$$h = \begin{pmatrix} \alpha_1 & \beta_1 \\ -\frac{a}{c}\beta_1 & \alpha_1 - \frac{b}{c}\beta_1 \end{pmatrix},$$

where (α_1, β_1) is a minimal positive integer solution of $(*)$.

Chapter 4

Sarkisov Program

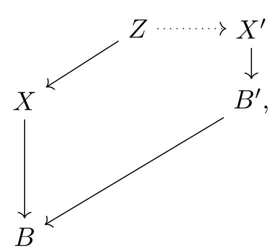
Recall form Section 2.2.3 that the outcomes of the MMP can be categorized into two types: Minimal models which consists in varieties X whose canonical class K_X is nef and Mori fiber spaces which are the ones with anticanonical class $-K_X$ relatively ample for an appropiate fibration. Mori fiber spaces are the outputs of the MMP when we start with a uniruled variety. Due to the various choices made when running the MMP, different outputs can arise starting with the same variety. Thus, it is natural to study the birational maps between these outputs within the same birational class. The *Sarkisov program* provides an algorithmic approach to the factorization of birational maps between Mori fiber spaces in terms of simpler birational maps, called *Sarkisov links*. It was established in dimension 3 in [Cor95], and in higher dimensions in [HM13].

4.1 Introduction

We start with the definition of the four types of Sarkisov links between Mori fiber spaces.

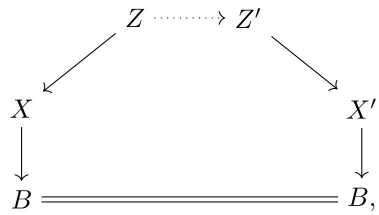
Definition 4.1.1 (Sarkisov links)**.** In the following diagrams, $X \to B$ and $X' \to B'$ denote Mori fiber spaces.

(I) A *Sarkisov diagram of type (I)* is a commutative diagram



where $Z \to X$ is a divisorial contraction and $Z \dashrightarrow X'$ is a sequence of flips, flops and antiflips. The map $X \dashrightarrow X'$ is called a *Sarkisov link of type (I)*.

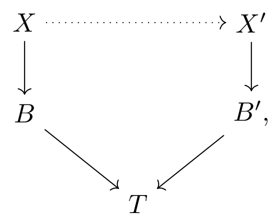
(II) A *Sarkisov diagram of type (II)* is a commutative diagram



where $Z \to X$ and $Z' \to X'$ are divisorial contractions and $Z \dashrightarrow Z'$ is a sequence of flips, flops and antiflips. In order to avoid trivial diagrams, we also require that the common relative effective cone of Z and Z' over B be generated by the exceptional divisors of $Z \to X$ and $Z' \to X'$. The map $X \dashrightarrow X'$ is called a *Sarkisov link of type (II).*

(III) A *Sarkisov link of type (III)* is the inverse of a Sarkisov link of type (I).

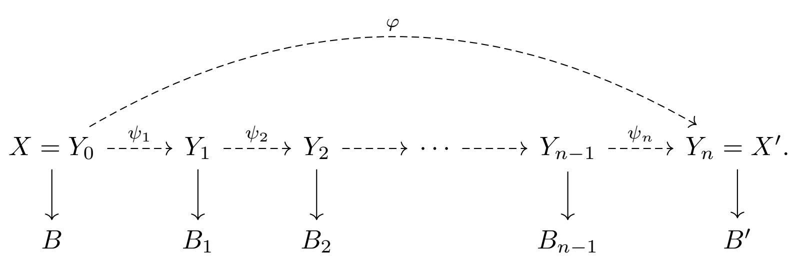
(IV) A *Sarkisov diagram of type (IV)* is a commutative diagram



where $X \dashrightarrow X'$ is a sequence of flips, flops and antiflips, and $B \to T$ and $B' \to T$ are Mori contractions. In order to avoid trivial diagrams, we also require that the common relative effective cone of X and X' over T be generated by the pullbacks to X and X' of ample divisors on B and B', respectively. The map $X \dashrightarrow X'$ is called a *Sarkisov link of type (IV).*

In the context of a Sarkisov diagram of type (I) or (II) above, we say that the divisorial contraction $Z \to X$ *initiates the Sarkisov link.*

Theorem 4.1.2 (The Sarkisov Program - [Cor95], [HM13])**.** Every birational map $\varphi\colon X \dashrightarrow X'$ between Mori fiber spaces X/B and X'/B' can be factorized as a composition of Sarkisov links:



Recall that for a variety X, the set of birational maps $X \dashrightarrow X$ forms a group, which is denoted by $\mathrm{Bir}(X)$. Thus, the Sarkisov program allows us to investigate $\mathrm{Bir}(X)$ when X is a Mori fiber space. Specifically, if X is a Fano variety with Picard rank $\rho(X) = 1$, the morphism $X \to \mathrm{Spec}(\mathbb{C})$ gives X the structure of a Mori fiber space. In particular, when $X = \mathbb{P}^n$, the Sarkisov program is an excellent tool for studying the Cremona group.

Definition 4.1.3 (Cremona group)**.** The *Cremona group* is the group $\mathrm{Bir}(\mathbb{P}^n)$ of all birational maps $\mathbb{P}^n \dashrightarrow \mathbb{P}^n$. It is sometimes denoted in the literature by $\mathrm{Cr}_n(\mathbb{C})$. Elements in $\mathrm{Bir}(\mathbb{P}^n)$ are called *Cremona transformations* of $\mathbb{P}^n$.

Some notable subgroups of $\mathrm{Bir}(X)$, for a given variety X, are as follows. Let $Z \subset X$ be a subvariety of X. We define the subgroup $\mathrm{Bir}(X; Z)$ of $\mathrm{Bir}(X)$ as the group of birational maps X that stabilize Z, i.e., those $\varphi \in \mathrm{Bir}(X)$ such that $\varphi(Z) = Z$. Similarly, we define $\mathrm{Aut}(X; Z) \subset \mathrm{Bir}(X; Z)$ as the group of elements in $\mathrm{Bir}(X; Z)$ that are regular maps. In the special case $X = \mathbb{P}^n$, the group $\mathrm{Bir}(\mathbb{P}^n; Z)$ is commonly referred to in the literature as the *Decomposition group* of Z, and sometimes denoted by $\mathrm{Dec}(Z)$.

Our primary focus is the case where $X = \mathbb{P}^3$ (or more generally, X is a Fano threefold with $\rho(X) = 1$) and S is a smooth quartic surface (or, more generally, $S \in |-K_X|$ is a smooth anticanonical surface). This will be further explored in Chapter 5. In fact, a first result from birational geometry that we will employ is the following consequence of the Sarkisov program. This result, due to Takahashi [Tak98, Theorem 2.3 and Remark 2.4], is adapted here in a manner suitable for investigating Gizatullin's problem and addressing broader contexts.

Proposition 4.1.4. Let Y be a smooth Fano 3-fold with Picard rank $\rho(Y) = 1$, $\mathrm{Pic}(Y) = \mathbb{Z}H_Y$ and index s (i.e., $-K_Y = sH_Y$), and $S \in |-K_Y|$ be a smooth surface. Assume that any irreducible reduced curve $C \subset S$ with $\deg(C) := C \cdot H_Y < s^2 H_Y^3$ is a complete intersection, i.e., $C = S \cap T$ for some hypersurface $T \subset Y$. Then $\mathrm{Bir}(Y; S) = \mathrm{Aut}(Y; S)$.

Proof. Assume by contradiction there exists $\varphi \in \mathrm{Bir}(Y) \setminus \mathrm{Aut}(Y)$ stabilizing S. First, note that the pair (Y, S) is canonical. Let $\mathcal{H}'$ be a very ample class on Y and $\mathcal{H}$ its proper transform under φ. Since φ is nonregular, $(Y, (1-\varepsilon)S + \varepsilon a\mathcal{H})$ is not canonical for $0 < \varepsilon < 1$ and $a > 0$ such that $K_Y + a\mathcal{H} \equiv 0$. This follows from the Noether-Fano inqualities of the Sarkisov program (see [Cor95, Theorem 4.2] and [Tak98, Theorem 1.4]) Let g be a resolution of indeterminancy of φ:

$$\begin{array}{ccc} & Z & \\ {}^{g}\swarrow & & \searrow^{g'} \\ Y & \overset{\varphi}{\dashrightarrow} & Y. \end{array}$$

Let $S_Z = g_*^{-1}S$, then $g : (Z, (1-\varepsilon)S_Z) \dashrightarrow (Y, (1-\varepsilon)S)$ and $g' : (Z, (1-\varepsilon)S_Z) \dashrightarrow (Y, (1-\varepsilon)S)$ satisfy conditions $(*)$ of [Tak98, Definition 1.2] for $0 < \varepsilon < 1$. Since $(Y, (1-\varepsilon)S + \varepsilon a\mathcal{H})$ is not canonical, there exists an exceptional divisor E of g such that $C = g(E) \subset S$ is a curve and the discrepancy of $K_Y + (1-\varepsilon)S + \varepsilon a\mathcal{H}$ is negative at E, for $0 < \varepsilon \ll 1$. At its generic point, we can view E as the exceptional divisor of the blowing up $\pi : W \to Y$ with center C. Let $\mathcal{H}_W$ the proper transform of $\mathcal{H}$ on W. Then $\mathcal{H}_W = \pi^*\mathcal{H} - mE$ with $ma > 1$. Thus,

$$\begin{aligned} 0 \le (\mathcal{H}_W)^2 \cdot H_Y &= (\pi^*\mathcal{H} - mE)^2 \cdot H_Y \\ &= \pi^*\mathcal{H} \cdot (\pi^*\mathcal{H} - mE) \cdot H_Y - mE \cdot (\pi^*\mathcal{H} - mE) \cdot H_Y \\ &= (\pi^*\mathcal{H})^2 \cdot H_Y - 2m\pi^*\mathcal{H} \cdot E \cdot H_Y + m^2E^2 \cdot H_Y \\ &= (\pi^*\mathcal{H})^2 \cdot H_Y - 2m\deg(C)\pi^*\mathcal{H} \cdot e + m^2\deg(C)E \cdot e \\ &= (\pi^*\mathcal{H})^2 \cdot H_Y - m^2\deg(C). \end{aligned}$$

Therefore,

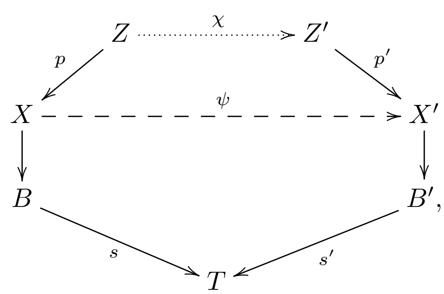
$$\deg C \leq \frac{(\pi^*\mathcal{H})^2 \cdot H_Y}{m^2} = \frac{s^2 H_Y^3}{a^2 m^2} < s^2 H_Y^3,$$

which implies that $C = S \cap T$, with T a surface on Y. Let $C_0 \subset Z$ be the proper transform of the curve $T_0 \cap T$ with T_0 a general surface on Y. Thus,

$$\begin{aligned} 0 \leq \mathcal{H}_W \cdot C_0 &= (\pi^*\mathcal{H} - mE) \cdot C_0 \\ &= \mathcal{H} \cdot T_0 \cdot T - mT_0 \cdot C \\ &= \frac{1}{a} S \cdot T_0 \cdot T - mT_0 \cdot C \\ &= \left(\frac{1}{a} - m\right) T_0 \cdot C \\ &< 0, \end{aligned}$$

where the last inequality follows from $am > 1$. Contradiction! □

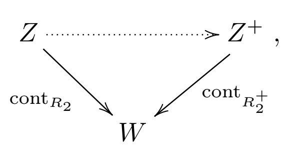
We return now to the general theory of the Sarkisov program. Notice that each type of Sarkisov link is recovered by a commutative diagram



where the possibilities for the maps are:

1. χ is a composition of antiflips/flops/flips or an isomorphism,
2. p and p' are divisorial contractions or isomorphisms,
3. s and s' extremal contractions or isomorphisms,
4. $\rho(Z/T) = \rho(Z'/T) = 2$.

The condition $\rho(Z/T) = 2$ implies that either p or s are isomorphism and similarly for $\rho(Z'/T) = 2$. Moreover, the Sarkisov link it determined by the two extremal rays R_1 and R_2 that generates $\overline{\mathrm{NE}}(Z/T) \subset \overline{\mathrm{NE}}(Z)$. This is known as the 2-*ray game*, and roughly speaking, the idea is as follows. If $\chi\colon Z \dashrightarrow Z'$ is an isomorphism, the map $Z \xrightarrow{p} X \to B \to T$ is given by the contraction of R_1 and the map $Z \xrightarrow{p'} X' \to B' \to T$ is given by the contraction of R_2. If χ is a composition of antiflips/flops/flips, either R_1 or R_2 gives a small contraction. Assume the contraction of R_2 is small; in this case, we have an antiflip/flop/flip diagram



where $\overline{\mathrm{NE}}(Z^+/T)$ is again a 2-dimensional cone generated by extremal rays R_1^+ and R_2^+ and the contraction of, say, R_2^+ is small. To continue with the "game", we then contract R_1^+. The process repeats until we end up with a variety Z' such that $\overline{\mathrm{NE}}(Z'/T)$ is generated by rays R_1' and R_2', where one of the two contractions is not small.

This motivates us to adopt the perspective presented in [BLZ21], where Sarkisov links are associated with rank 2 fibrations. Using this approach, they managed to show, for instance, that the Cremona group $\mathrm{Bir}(\mathbb{P}^n)$ is not simple. We refer to [BLZ21, Definition 2.2.] for the definition of relative Mori Dream Space.

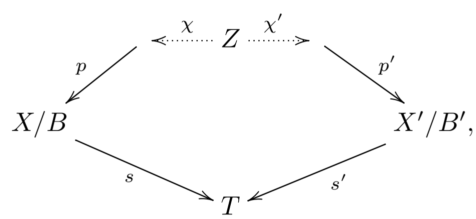
Definition 4.1.5 ([BLZ21, Definition 3.1])**.** Let r be an integer. A morphism $\eta\colon X \to B$ is a *rank r fibration* if the following conditions hold:

1. X/B is a relative Mori Dream Space.
2. $\dim(X) > \dim(B)$ and $\rho(X/B) = r$.
3. X is $\mathbb{Q}$-factorial and terminal, and for any divisor D on X, the output of any D-MMP from X over B is still $\mathbb{Q}$-factorial and terminal.
4. There exists an effective $\mathbb{Q}$-divisor Δ_B on B such that (B, Δ_B) is klt.
5. $-K_X$ is η-big.

The notion of rank r fibrations encompasses the notions of Mori fiber spaces and Sarkisov links. In particular, a rank 1 fibration is a Mori fiber space, while rank 2 fibrations correspond to Sarkisov links.

Lemma 4.1.6 ([BLZ21, Lemma 3.3])**.** Let $\eta\colon X \to B$ be a surjective morphism between normal varieties. Then X/B is a rank 1 fibration if and only if X/B is a Mori fiber space.

Lemma 4.1.7 ([BLZ21, Lemma 3.7])**.** Let Z/T be a rank 2 fibration. Then Z/T factorizes through exactly two rank 1 fibrations X/B and X'/B', fitting into the diagram



where the maps χ and χ' are sequences of flips, flops or antiflips, and p, p', s, s' are morphisms of relative Picard rank 1 or isomorphisms.

For the purposes of the Sarkisov program, it is important to have a classification of Sarkisov links. In Section 4.3, we focus on the case where the Sarkisov link involves the Mori fiber space $Y \to \mathrm{Spec}(\mathbb{C})$, where Y is a Fano 3-fold with $\rho(Y) = 1$ (see Example 2.2.23), particularly when $Y = \mathbb{P}^3$. Specially, $Y \to \mathrm{Spec}(\mathbb{C})$ appears in the source of Sarkisov links of type (I) or (II). Consequently, the Sarkisov link necessarily involves a divisorial contraction $p\colon X \to Y$, which can be either a blow-up of a curve or a weighted blow-up of a point (see [Tzi03, Proposition 1.2] and [Kaw01, Theorem 1.1], respectively).

Regarding Gizatullin's problem, we will concentrate on the former case. This choice is motivated by the *volume-preserving* version of the Sarkisov program, which imposes restrictions on the birational operations allowed in *volume preserving* Sarkisov links. We develop these notions in the next section.

4.2 Calabi-Yau pairs and volume preserving birational maps

In this section we introduce a special version of the Sarkisov program, which is a generalization of the original framework for the setting when the varieties and maps have an additional structure: *Mori fibered Calabi-Yau pairs* and *volume preserving Sarkisov links.* It fits perfectly with the study of Gizatullin's problem since the pair $(\mathbb{P}^3, S)$ is a Calabi-Yau pair when $S \subset \mathbb{P}^3$ is a smooth quartic, and a Cremona transformation $\varphi \in \mathrm{Bir}(\mathbb{P}^3)$ belongs to $\mathrm{Bir}(\mathbb{P}^3; S)$ if and only if it is volume preserving.

We follow the notions and framework developed in [CK16] and [ACM23].

Definition 4.2.1 (Calabi-Yau pair)**.** A *Calabi-Yau pair* is a pair (X, D) consisting of a terminal projective $\mathbb{Q}$-factorial variety X and an effective Weil divisor D on X such that $K_X + D \sim 0$ and (X, D) has log canonical singularities. We say that a Calabi-Yau pair (X, D) is *canonical* if it has canonical singularities.

Remark 4.2.2. The condition that $K_X + D \sim 0$ implies that there exist a top rational differential form $\omega_{X,D} \in H^0(X, \Omega^n_X)$, unique up to scalar multiplication, such that $D + \mathrm{div}(\omega_{X,D}) = 0$. We call $\omega_{X,D}$ the *volume form* associated to the Calabi-Yau pair (X, D).

Example 4.2.3. (Weak Fano varieties) Let X be a smooth variety variety. We say that X is *weak Fano* if the anticaconical divisor $-K_X$ is nef and big. Thus any Fano variety is a weak Fano varieties. A weak Fano variety X is said to be *strictly weak Fano* if it is not Fano. Consider a reduced divisor $D \in |-K_X|$. Moreover, the pair (X, D) is a Calabi-Yau pair. In particular, when $S \subset \mathbb{P}^3$ is a smooth quartic surface, the pair $(\mathbb{P}^3, S)$ is a canonical Calabi-Yau pair.

Definition 4.2.4 (Volume preserving maps)**.** Let (X, D_X) and (Y, D_Y) be Calabi-Yau pairs, and $f\colon X \dashrightarrow Y$ a birational map, inducing an identification of the function fields $\mathbb{C}(X) \cong_{\mathbb{C}} \mathbb{C}(Y)$. We say that f is *volume preserving with respect to* (X, D_X) *and* (Y, D_Y) if, for every divisorial valuation E of $\mathbb{C}(X) \cong_{\mathbb{C}} \mathbb{C}(Y)$, the discrepancies of E with respect to the pairs (X, D_X) and (Y, D_Y) are equal: $a(E, X, D_X) = a(E, Y, D_Y)$.

Definition 4.2.5. Let (X, D) be a Calabi-Yau pair. Clearly, a composition of volume preserving birational self-maps with respect to (X, D) is volume preserving. Thus, we define $\mathrm{Bir}^{v.p.}(X, D)$ as the group of volume preserving birational self-maps with respect to (X, D).

We point out that the group $\mathrm{Bir}^{v.p.}(X, D)$ is a subgroup of the group $\mathrm{Bir}(X)$ of birational self-maps of X.

Remark 4.2.6. The *volume preserving* terminology is explained by the following characterization (see [CK16, Remark 1.7]). Given Calabi-Yau pairs (X, D_X) and (Y, D_Y), consider the unique (up to scaling) associated volume forms ω_{X,D_X} of (X, D_X) and ω_{Y,D_Y} of (Y, D_Y). A birational map $f : X \dashrightarrow Y$ induces an identification of the spaces of rational volume forms on X and Y. It is volume preserving with respect to (X, D_X) and (Y, D_Y) if and only if it identifies the rational volume forms ω_X and ω_Y, up to scaling, i.e., $f^*(\omega_{Y,D_Y}) = \lambda\omega_{X,D_X}$ for some $\lambda \in \mathbb{C}^*$.

Proposition 4.2.7 ([ACM23, Proposition 2.6])**.** Let (X, D_X) and (Y, D_Y) be canonical Calabi-Yau pairs, and $f \colon X \dashrightarrow Y$ a birational map. Then $f \colon X \dashrightarrow Y$ is volume preserving with respect to (X, D_X) and (Y, D_Y) if and only if it restricts to a birational map between D_X and D_Y.

Hence, if we consider the volume preserving self-maps of a given canonical Calabi-Yau pair (X, D), the restriction defines a homomorphism of groups

$$\Psi \colon \operatorname{Bir}^{v,p}(X, D) \to \operatorname{Bir}(D).$$

Remark 4.2.8. In the particular case when $S \subset \mathbb{P}^3$ is a smooth quartic surface, the pair $(\mathbb{P}^3, S)$ is canonical and so the group of Cremona transformations stabilizing S coincides with the group of volume preserving Cremona transformations with respect to $(\mathbb{P}^3, S)$:

$$\operatorname{Bir}\left(\mathbb{P}^3; S\right) \; = \; \operatorname{Bir}^{v.p.}\left(\mathbb{P}^3, S\right).$$

Moreover, since $\operatorname{Bir}(S) = \operatorname{Aut}(S)$, Problem 1 is equivalent to asking if the restriction homomorphism

$$\Psi \colon \operatorname{Bir}(\mathbb{P}^3; S) \to \operatorname{Aut}(S)$$

is surjective. More generally, one may ask for its image. In Section 5.2 we revisit this perspective to investigate Gizatullin's problem.

Definition 4.2.9 (Mori fibered Calabi-Yau pair)**.** A *Mori fibered Calabi-Yau pair* is a Calabi-Yau pair (X, D), together with a Mori fiber space structure $X \to B$.

An important tool to study volume preserving birational maps between Calabi-Yau pairs is the volume preserving variant of the Sarkisov program established in [CK16]. Before we state it, we introduce the volume preserving version of Sarkisov links.

Definition 4.2.10 ([CK16, Definition 1.12 and Remark 1.13])**.** A *volume preserving Sarkisov link* is a Sarkisov link as in Definition 4.1.1, with the following additional data and property.

- There are effective Weil divisors D_X on X, $D_{X'}$ on X', D_Z on Z, and $D_{Z'}$ on Z', making the pairs (X, D_X), $(X', D_{X'})$, (Z, D_Z) and $(Z', D_{Z'})$ Calabi-Yau pairs.
- All the divisorial contractions, flips, flops and antiflips that constitute the Sarkisov link are volume preserving for these Calabi-Yau pairs.

Theorem 4.2.11 (Volume preserving Sarkisov Program - [CK16])**.** Every volume preserving birational map between Mori fibered Calabi-Yau pairs can be factorized as a composition of volume preserving Sarkisov links.

The volume preserving condition imposes strong restrictions on the links appearing in a Sarkisov decomposition. For example, in the context of Gizatullin's problem, we have the following.

Proposition 4.2.12. Let $S \subset \mathbb{P}^3$ be a smooth quartic surface, and $f\colon (X, D_X) \to (\mathbb{P}^3, S)$ a volume preserving divisorial contraction. Then $f\colon X \to \mathbb{P}^3$ is the blowup of a curve contained in S.

Proof. By [ACM23, Proposition 3.1], the center of the divisorial contraction $f\colon X \to \mathbb{P}^3$ is a curve $C \subset S$. By [Tzi03, Proposition 1.2], $f\colon X \to \mathbb{P}^3$ is the blowup of $\mathbb{P}^3$ along C. □

4.3 Classification of Sarkisov links from $\mathbb{P}^3 \to \mathrm{Spec}(\mathbb{C})$

In this section, we investigate when the blowup of $\mathbb{P}^3$ along a curve initiates a Sarkisov link.

Let $C \subset \mathbb{P}^3$ be a smooth curve and denote by $p\colon X \to \mathbb{P}^3$ the blowup of $\mathbb{P}^3$ along C. By Proposition 2.2.6, $N^1(X)$ and $N_1(X)$ have rank two. We will denote by H the class of the pullback of a general hyperplane in $\mathbb{P}^3$ and by E the excepcional divisor. These two classes generate $N^1(X)$. Furthermore, we denote by l the class of the pullback of a general line in $\mathbb{P}^3$ and by e the class of a fiber of E, i.e., a fiber of p over a point on C. These two classes generate $N_1(X)$. Thus, the perfect paring $N^1(X)_{\mathbb{R}} \times N_1(X)_{\mathbb{R}} \to \mathbb{R}$ is determined by

$$H \cdot l = 1, \quad H \cdot e = E \cdot l = 0, \quad E \cdot e = -1.$$

The relative Mori cone $\overline{\mathrm{NE}}(X/\mathrm{Spec}(\mathbb{C}))$ of $X \to \mathbb{P}^3 \to \mathrm{Spec}(\mathbb{C})$ is clearly the whole Mori cone $\overline{\mathrm{NE}}(X)$, which is two dimensional. One of its generators is e. The anticanonical divisor is $-K_X = p^*(-K_{\mathbb{P}^3}) - E = 4H - E$. Since $-K_X \cdot e = (4H - E) \cdot e = 1 > 0$, the extremal ray of $\overline{\mathrm{NE}}(X)$ generated by e is K_X-negative and its contraction gives exactly $p\colon X \to \mathbb{P}^3$.

Assuming that $p\colon X \to \mathbb{P}^3$ initiates a Sarkisov link, it could be of type (I) or (II) and it is determined by the contraction of the other extremal ray R of $\overline{\mathrm{NE}}(X)$. This contraction could be a Mori fiber space, divisorial or small. In the latter case, the link would proceed with a flip, flop or antiflip. Thus, we have

- The contraction of R gives a Mori fiber space, a divisorial contraction or a flip $\Leftrightarrow$ R is a K_X-negative extremal ray $\Leftrightarrow$ X is Fano.
- The contraction gives a flop $\Leftrightarrow$ R intersects K_X trivially $\Leftrightarrow$ X is strictly weak Fano.
- The contraction gives an antiflip $\Leftrightarrow$ R is a K_X-positive extremal ray $\Leftrightarrow$ X is not weak Fano.

Hence, we proceed naturally to analyze when the blowup X of $\mathbb{P}^3$ along a curve is weak Fano or not, and also, assuming one of these two possibilitites, when $p\colon X \to \mathbb{P}^3$ initiates a Sarkisov link. We start with some constraints on curves $C \subset \mathbb{P}^3$ whose blowup is weak Fano.

Proposition 4.3.1 ([BL15, Proposition 1.9])**.** Let $C \subset \mathbb{P}^3$ be a smooth curve of genus g and degree d. Denote by X the blow-up of $\mathbb{P}^3$ along C and E the exceptional divisor. Then

1. $K_X^2 \cdot E = -K_{\mathbb{P}^3} \cdot C = 4d + 2 - 2g$,

2. $(-K_X)^3 = 64 - 8d - 2 + 2g$,
3. $(-K_X)^3 > 0$ if and only if $4d + 30 \leq g$.

Moreover, if X is weak Fano it holds that

4. $\dim |-K_X| = \frac{1}{2}(-K_X)^3 + 2 \geq 3$,
5. $4d + 30 \leq g$.

The following criterion to determine whether the blowup of $\mathbb{P}^3$ along a curve C initiates a Sarkisov link. This is a special case of Lemma 4.1.7.

Lemma 4.3.2. Let $C \subset \mathbb{P}^3$ be a curve, and let X denote the blowup of $\mathbb{P}^3$ along C. Then $X \to \mathbb{P}^3$ initiates a Sarkisov link if and only if $X \to \mathbb{P}^3 \to \mathrm{Spec}(\mathbb{C})$ is a rank 2 fibration.

Now, we provide a degree bound:

Lemma 4.3.3. Let $C \subset \mathbb{P}^3$ be a curve, and $X \to \mathbb{P}^3$ the blowup of $\mathbb{P}^3$ along C. If $X \to \mathrm{Spec}(\mathbb{C})$ is a rank 2 fibration, then $\deg(C) < 16$.

Proof. We denote by E the exceptional divisor of the blowup $X \to \mathbb{P}^3$, and by H the pullback of the hyperplane class of $\mathbb{P}^3$. By definition of a rank 2 fibration, $-K_X = 4H - E$ is big. We take n sufficiently large so that $|-nK_X|$ has no base components, and $-nK_X \sim A+F$, with A very ample and F effective. For any $T \in |-nK_X|$, we denote by $\check{T}$ its pushforward to $\mathbb{P}^3$. Note that $\check{T}$ is a surface of degree $4n$ containing C with multiplicity at least n. Let $T_1, T_2 \in |-nK_X|$ be general members. We claim that $T_1 \cap T_2 \not\subset E$. Indeed, if $T_1 \in |-nK_X|$ and $D \in |A|$ are general members, then $E \cap T_1 \cap D$ consists of finitely many points, and so $T_1 \cap D \not\subset E$. If we take $T_2' = D + F \in |-nK_X|$, then $T_1 \cap T_2' \not\subset E$. This proves the claim. Therefore, as a 1-cycle,

$$\check{T}_1 \cdot \check{T}_2 = n^2 C + C',$$

with C' a nonzero effective cycle. By Bézout's Theorem, $\deg\left(\check{T}_1 \cdot \check{T}_2\right) = 16n^2$, and thus

$$16n^2 = \deg\left(\check{T}_1 \cdot \check{T}_2\right) = \deg(C)n^2 + \deg(C') > \deg(C)n^2.$$

Hence, $\deg(C) < 16$. □

In [BL12, Theorem 1.1], Blanc and Lamy classified smooth curves $C \subset \mathbb{P}^3$ whose blowups X are weak Fano. The following observation allows one to check which of these blowups give rank 2 fibrations.

Lemma 4.3.4. Let $C \subset \mathbb{P}^3$ be a curve, and suppose that the blowup X of $\mathbb{P}^3$ along C is terminal and weak Fano. Then $X \to \mathbb{P}^3 \to \mathrm{Spec}(\mathbb{C})$ is a rank 2 fibration if and only if the morphism to the anti-canonical model of X is either an isomorphism or a small contraction.

Proof. Since X is weak Fano, it is a Mori Dream Space and $-K_X$ is semi-ample. So conditions (1), (2), (4) and (5) in Definition 4.1.5 are all satisfied. If X is Fano, then any D-MMP is also a (K_X)-MMP. Thus, the output of any D-MMP is terminal, and so $X \to \mathbb{P}^3 \to \operatorname{Spec}(\mathbb{C})$ is a rank 2 fibration. From now on, suppose that X is strictly weak Fano. The morphism $X \to \check{X}$ to the anti-canonical model of X is either a divisorial contraction or a small contraction. When $X \to \check{X}$ is divisorial, $\check{X}$ has worse than terminal singularities, and therefore $X \to \mathbb{P}^3 \to \operatorname{Spec}(\mathbb{C})$ is *not* a rank 2 fibration, as it violates condition (3) in Definition 4.1.5. Suppose that $X \to \check{X}$ is a small contraction, and consider its flop

$$\begin{array}{ccc} X & \dashrightarrow & X^+ \\ & \searrow \quad \swarrow & \\ & \check{X}. & \end{array}$$

Note that $-K_{X^+}$ is the pullback of $-K_{\check{X}}$, which is ample, and so X^+ is again terminal and weak Fano. Write R_1^+ and R_2^+ for the two extremal rays of $\overline{\mathrm{NE}}(X^+)$, where R_1^+ corresponds to $X^+ \to \check{X}$ and R_2^+ is K_{X^+}-negative. Let D be any divisor on X. Then any D-MMP either ends with $\mathbb{P}^3$, or it factors through X^+. In the latter case, any further step is associated to the contraction of R_2^+, and is therefore a (K_{X^+})-MMP too. In any case, the output of any D-MMP is terminal, and so $X \to \mathbb{P}^3 \to \operatorname{Spec}(\mathbb{C})$ is a rank 2 fibration. □

The possibilities for the genus and degree of smooth curves $C \subset \mathbb{P}^3$ whose blowups X are weak Fano are listed in [BL12, Table 1], as well as whether or not the morphism $X \to \check{X}$ to the anti-canonical model is divisorial for a general curve in the corresponding Hilbert scheme. So, by putting together [BL12, Theorem 1.1 and Table 1] and Lemma 4.3.4, we get the following classification.

Theorem 4.3.5. Let $C \subset \mathbb{P}^3$ be a smooth curve of genus g and degree d, and let X denote the blowup of $\mathbb{P}^3$ along C. Suppose that X is weak Fano and $X \to \mathbb{P}^3 \to \operatorname{Spec}(\mathbb{C})$ is a rank 2 fibration. Then

$$(g,d) \in \left\{ \begin{array}{c} (0,1), (0,2), (0,3), (0,4), (0,5), (0,6), (0,7), (1,3), (1,4), (1,5), (1,6), (1,7), \\ (2,5), (2,6), (2,7), (\mathbf{2},\mathbf{8}), (\mathbf{3},\mathbf{6}), (3,7), (\mathbf{3},\mathbf{8}), (4,6), (4,7), (\mathbf{4},\mathbf{8}), (5,7), (\mathbf{5},\mathbf{8}), \\ (6,8), (\mathbf{6},\mathbf{9}), (7,8), (7,9), (8,9), (9,9), (10,9), (\mathbf{10},\mathbf{10}), (\mathbf{11},\mathbf{10}), (\mathbf{14},\mathbf{11}) \end{array} \right\}. \qquad (\dagger)$$

Conversely, suppose that $(g,d) \in (\dagger)$ and the smooth curve C satisfies the following conditions, which define an open subset of the Hilbert scheme $\mathcal{H}_{g,d}$ of curves of arithmetic genus g and degree d:

1. C does not admit 5-secant lines, 9-secant conics, nor 13-secant twisted cubics;
2. there are finitely many irreducible curves in X intersecting $-K_X$ trivially.

Then X is weak Fano and $X \to \operatorname{Spec}(\mathbb{C})$ is a rank 2 fibration.

The pairs $(\mathbf{g}, \mathbf{d})$ in bold are the ones that will be relevant to our approach to Gizatullin's problem in Section 5.2.

4.3.1 Sarkisov links centered on curves on quartic surfaces

In this subsection, we give some constraints on curves $C \subset \mathbb{P}^3$ whose blowups initiate Sarkisov links. While these curves are not completely classified in general, our main result is a classification of curves contained in smooth quartic surfaces with Picard rank 2 whose blowups initiate Sarkisov links (Proposition 4.3.11). Recall from Lemma 4.3.2

that the blowup $X \to \mathbb{P}^3$ of a curve $C \subset \mathbb{P}^3$ initiates a Sarkisov link if and only if $X \to \operatorname{Spec}(\mathbb{C})$ is a rank 2 fibration.

We first exclude curves that are complete intersections.

Lemma 4.3.6. Let $S \subset \mathbb{P}^3$ be a quartic surface, $C = S \cap T$ the complete intersection of S with another surface $T \subset \mathbb{P}^3$, and $X \to \mathbb{P}^3$ the blowup of $\mathbb{P}^3$ along C. Then $X \to \operatorname{Spec}(\mathbb{C})$ is not a rank 2 fibration.

Proof. Let $\Gamma \subset T$ be a general complete intersection curve of T with an hyperplane of $\mathbb{P}^3$. Thus, Γ meets C transversely at $S \cdot \Gamma = 4\deg(\Gamma)$ distinct points. Denote by $\tilde{\Gamma} \subset X$ its strict transform. Then $-K_X \cdot \tilde{\Gamma} = 0$. It follows from [Zik23b, Proposition 3.15] that $X \to \operatorname{Spec}(\mathbb{C})$ is not a rank 2 fibration. □

When we consider an arbitrary curve C contained in a surface $S \subset \mathbb{P}^3$, it is useful to be able to replace C with a better curve $C' \subset S$ that is linearly equivalent to C in S. For instance, we may want to take C' smooth. The next results allow us to compare the blowup of $\mathbb{P}^3$ along C with the blowup of $\mathbb{P}^3$ along C'.

Lemma 4.3.7. Let S be a smooth quartic surface, and $C, C' \subset S$ curves that are linearly equivalent in S. Denote by $X \to \mathbb{P}^3$ and $X' \to \mathbb{P}^3$ the blowups of $\mathbb{P}^3$ along C and C', and by $\tilde{S}$ and $\tilde{S}'$ the strict transforms of S on X and X', respectively. For any curve $\check{\Gamma} \subset S$, denote by Γ and Γ' its image in $\tilde{S}$ and $\tilde{S}'$ under the identifications $S \cong \tilde{S}$ and $S \cong \tilde{S}'$ respectively. Assume that X and X' are $\mathbb{Q}$-Gorenstein. Then the following hold:

1. $(-K_X) \cdot \Gamma = (-K_{X'}) \cdot \Gamma'$.
2. X is weak Fano if and only if so is X'.
3. Suppose that X is weak Fano and $X \to \operatorname{Spec}(\mathbb{C})$ is a rank 2 fibration. If C' is general in $|C|$, then X' is weak Fano and $X' \to \operatorname{Spec}(\mathbb{C})$ is a rank 2 fibration.

Proof. We denote by E and E' the exceptional divisors of $X \to \mathbb{P}^3$ and $X' \to \mathbb{P}^3$, respectively. We abuse notation and use the same symbol H to denote the hyperplane class in $\mathbb{P}^3$ and its pullbacks to X and X'. We shall see that, for any $a, b \in \mathbb{Z}$, $(aH - bE) \cdot \Gamma = (aH - bE') \cdot \Gamma'$. This yields (1) as a special case.

$$\begin{aligned}(aH - bE) \cdot \Gamma = (aH - bE)|_{\tilde{S}} \cdot \Gamma = (aH|_S - bC) \cdot \check{\Gamma} \\ = (aH|_S - bC') \cdot \check{\Gamma} = (aH - bE')|_{\tilde{S}'} \cdot \Gamma' = (aH - bE') \cdot \Gamma'.\end{aligned}$$

If $\Gamma \subset X$ is any curve with $(-K_X) \cdot \Gamma < 0$, then $\Gamma \subset \tilde{S}$. By (1) $(-K_{X'}) \cdot \Gamma' < 0$. By the symmetric nature of the argument, we get that $-K_X$ is nef if and only if so is $-K_{X'}$. Similarly, for bigness, we have

$$(-K_X)^3 = (-K_X|_{\tilde{S}})^2 = (4H|_{\tilde{S}} - C)^2 = (4H|_{\tilde{S}'} - C')^2 = (-K_{X'}|_{\tilde{S}'})^2 = (-K_{X'})^3$$

In particular $(-K_X)^3 > 0$ if and only if $(-K_{X'})^3 > 0$. This gives (2).

For (3), assume that X is weak Fano and $X \to \operatorname{Spec}(\mathbb{C})$ is a rank 2 fibration. Choosing $C' \in |C|$ smooth and using (2) and Lemma 4.3.4, it suffices to prove that, for general C', the morphism to the anti-canonical model of X' is small. For that, we perform a blowup in a family over the base $|C|$: set $\mathcal{P} := |C| \times \mathbb{P}^3$, $\mathcal{Z} := \{(D,p) \in |C| \times \mathbb{P}^3 \,|\, p \in D\}$, and denote by $\mathcal{X}$ the blowup of $\mathcal{P}$ along $\mathcal{Z}$. For a general element $b \in |C|$, the fiber $\mathcal{X}_b$ is weak Fano, and so $-K_{\mathcal{X}_b}$ is semiample. Therefore, the relative anti-canonical map over $|C|$ is a morphism on the preimage $\mathcal{U}$ of an open subset of $|C|$. Denote by $\mathcal{E}$ the closure of the exceptional locus of the morphism to the relative anti-canonical model on $\mathcal{U}$. By assumption, the fiber of $\mathcal{E}$ over $C \in |C|$ is at most one dimensional. So, by upper-semicontinuity of the dimension of the fiber, the fiber of $\mathcal{E}$ over a general point $C' \in |C|$ is also at most one dimensional, i.e., the morphism to the anti-canonical model of X' is small. □

By Theorem 4.3.5, if $C \subset \mathbb{P}^3$ is a smooth curve satisfying the generality conditions (1) and (2) of Theorem 4.3.5, its blowup is weak Fano and initiates a Sarkisov link. Condition (1) ensures that the blow-up is weak Fano, while Condition (2) ensures that it initiates a Sarkisov link. The next two propositions establish criteria to verify these conditions when the curve C lies on a smooth quartic surface $S \subset \mathbb{P}^3$. In what follows, we slightly abuse notation by using the same symbol H to denote the hyperplane class in $\mathbb{P}^3$, the class of its pullback on X, and its restriction class to S.

Proposition 4.3.8. Let $S \subset \mathbb{P}^3$ be a smooth quartic surface and $C \subset S$ be a smooth curve of genus g and degree d. Denote by X the blow-up of $\mathbb{P}^3$ along C. Then the following holds:

1. $-K_X$ is nef if and only if $4H - C$ is nef on S.
2. X is weak Fano if and only if $4H - C$ is nef and big on S.
3. If X is weak Fano, then the linear system $|-K_X|$ is base point free on X if and only if the linear system $|4H - C|$ is base point free on S.

Proof. Let $\tilde{S}$ be the strict transform of S under the blowup $p\colon X \longrightarrow \mathbb{P}^3$. Denote by E the exceptional divisor of p. Thus $-K_X = 4H - E$, $S \cong \tilde{S}$,

$$(-K_X)|_{\tilde{S}} = (4H - E)|_{\tilde{S}} = 4H - C \quad \text{and} \quad (-K_X)^3 = (-K_X|_{\tilde{S}})^2 = (4H - C)^2. \tag{4.1}$$

Since $\tilde{S} \cong S$, one has a one-to-one correspondence between curves on $\tilde{S}$ and curves on S given by taking the image (in one direction) and the strict transform (in the other direction) under p. Moreover, let $\Gamma \subset \tilde{S} \subset X$ be a curve and denote by $\check{\Gamma} \subset S$ its image. Thus, it follows from (4.1) that

$$-K_X \cdot \Gamma = (-K_X)|_{\tilde{S}} \cdot \Gamma = (4H - C) \cdot \check{\Gamma}.$$

Since any curve on X intersecting $-K_X$ negatively is contained in $\tilde{S}$, statement (1) holds. Assertion (2) follows from (1) and (4.1).

To prove (3), we observe that from the fundamental exact sequence associated to $\tilde{S}$ and the identification of $\tilde{S} \cong S$ we have the following sequence in cohomology

$$0 \longrightarrow H^0(X, \mathcal{O}_X(-K_X - S)) \longrightarrow H^0(X, \mathcal{O}_X(-K_X)) \longrightarrow H^0(S, \mathcal{O}_S(4H - C)) \longrightarrow H^1(X, \mathcal{O}_X(-K_X - S)) \cdots .$$

Since $\mathcal{O}_X(-K_X - S) \cong \mathcal{O}_X$, we get that $H^1(X, \mathcal{O}_X(-K_X - S)) = 0$ and so the map $H^0(X, \mathcal{O}_X(-K_X)) \longrightarrow H^0(S, \mathcal{O}_S(sH_S - C))$ is surjective. Therefore, the linear system $|-K_X|$ has base points on X if and only if $|4H - C|$ has base points on S. □

Proposition 4.3.9. Let S, C and X be as in Proposition 4.3.8. Assume that C is not a complete intersection of S with another hypersurface of $\mathbb{P}^3$, $4H-C$ is ample and its linear system is base point free on S. Then X is weak Fano and there exists finitely many curves intersecting K_X trivially (i.e., the morphism to the anticanonical model is small).

Proof. By assumptions and Proposition 4.3.8(2) and (3), the blowup X is weak Fano and $|-K_X|$ is base point free. Now, assume that the morphism $X \to Z$ to the anticanonical model is not small, then it is divisorial. Denote by $D \in \mathrm{Pic}(X)$ the divisor contracted by it. Then $D = aH - bE$ and

$$0 = (-K_X)^2 \cdot D = (-K_X \cdot D)|_{\tilde{S}} = \check{D}|_S \cdot (4H - C) = (aH - bC) \cdot (4H - C),$$

where $\check{D}$ is the pushforward of D under the blowup $p \colon X \to \mathbb{P}^3$. We get that $aH - bC = 0$ by the ampleness of $4H - C$. This implies that C is a complete intersection of S with another hypersurface of $\mathbb{P}^3$ and contradicts the hypothesis. Thus, $a = 0 = b$ and so $D = 0$, a contradiction! Therefore we get the assertion. □

In particular, under the assumptions of Proposition 4.3.9, Lemma 4.3.4 guarantees that the blow-up $p \colon X \to \mathbb{P}^3$ initiates a Sarkisov link. For the specific case of smooth quartic surfaces with Picard rank 2, the preceding propositions yield the following corollary, which establishes a necessary condition for the blow-up of a curve $C \subset S$ to initiate a Sarkisov link.

Corollary 4.3.10. Let $S \subset \mathbb{P}^3$ be a smooth quartic surface with $\rho(S) = 2$ and $C \subset \mathbb{P}^3$. Denote by X the blowup of $\mathbb{P}^3$ along C. Assume that $\mathrm{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}C$, $r(S) \neq 9$ and $4H - C$ is ample on S. Then X is weak Fano and yields a Sarkisov link.

Proof. By ampleness of $4H - C$ and Proposition 4.3.8(2), X is weak Fano. Moreover, $|4H - C|$ is base point free since $4H - C$ generates $\mathrm{Pic}(S)$ and by Proposition 3.5.2. Furthermore, the blowup $p \colon X \to \mathbb{P}^3$ along C initiates a Sarkisov link φ_C by Proposition 4.3.8(3) and Proposition 4.3.9. □

We end this section with a classification of curves contained in smooth quartic surfaces with Picard rank 2 whose blowups initiate Sarkisov links.

Proposition 4.3.11. Let $C \subset \mathbb{P}^3$ be a (possibly singular) curve of arithmetic genus p_a and degree d lying on a smooth quartic surface S with Picard rank 2. Let X be the blowup of $\mathbb{P}^3$ along C, and suppose that $X \to \mathrm{Spec}(\mathbb{C})$ is a rank 2 fibration. Then X is weak Fano and (p_a, d) is one of the pairs in the list (†) of Theorem 4.3.5.

Proof. We denote by S a smooth quartic surface with Picard rank 2 containing C, and by $\tilde{S}$ its strict transform on X. Suppose that $X \to \mathrm{Spec}(\mathbb{C})$ is a rank 2 fibration. We will show that X is weak Fano. Suppose that X is not weak Fano. Then the Sarkisov link initiated by $X \to \mathbb{P}^3$ proceeds with an anti-flip. By Lemma 2.2.30(3), the extremal ray corresponding to the associated small contraction of X is generated by a smooth rational curve $\Gamma \subset X$ such that $\tilde{S} \cdot \Gamma = -K_X \cdot \Gamma < 0$. In particular, $\Gamma \subset \tilde{S}$. By [ACM23, Lemma 4.4], we must have $-K_X \cdot \Gamma = -1$. Denote by $\check{\Gamma}$ the image of Γ in $\mathbb{P}^3$. Since $X \to \mathrm{Spec}(\mathbb{C})$ is a rank 2 fibration, Lemma 4.3.6 implies that C is not a complete intersection. As in (3.2), we assume that $\mathrm{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}W$, where H is a hyperplane section of S, the intersection product is given by the matrix (3.2) and S has discriminant $r(S) = b^2 - 8c > 0$. By changing W if necessary, we may assume that $0 < b < 16$. Write $\check{\Gamma} = \alpha H + \beta W$ and $C = \delta H + \gamma W$ in $\mathrm{Pic}(S)$ with $\alpha, \beta, \delta, \gamma \in \mathbb{Z}$. The conditions that $\check{\Gamma}$ is a rational curve and $-K_X \cdot \Gamma = -1$ give the following system:

$$\begin{cases} 0 < H \cdot \check{\Gamma} = 4\alpha + b\beta \\ -2 = \check{\Gamma}^2 = 4\alpha^2 + 2b\alpha\beta + 2c\beta^2 \\ -1 = (4H - C) \cdot \check{\Gamma} = (16 - d)\,\alpha + \left(4b - \delta b + \frac{d^2 - 8(p_a - 1) - \gamma^2 b^2}{4\gamma}\right)\beta. \end{cases} \tag{4.2}$$

Note that since C is not a complete intersection, the sublattice $\langle H, C\rangle$ of $\mathrm{Pic}(S)$ has rank two and signature $(1, 1)$. Thus, $d^2 - 8(p_a - 1) = -\operatorname{disc}(H, C) > 0$. It follows that $p_a \leq d^2/8$, and $d < 16$, by Lemma 4.3.3 Moreover, from $d^2 - 8(p_a - 1) = r(S)\gamma^2$ and the fact that $d < 16$ and $p_a \geq 0$, it follows that $r(S), \gamma^2 \leq 233$. Therefore, there are finitely many possibilities for the pair (p_a, d), and hence for the integers b, c, δ and γ. Using the Matlab code below, one may verify that subject to these conditions, the system (4.2) admits an integer solution (α, β) if and only if $(p_a, d) = (15, 11)$. In this case, the discriminant of S is $r(S) = 9$ and $\gamma = \pm 1$. So $\mathrm{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}C$, and the integer solution of the system (4.2) gives a line $\ell = \check{\Gamma} \sim 3H - C$ contained in S. We will show that the rational contraction $X \dashrightarrow Y$ to the anti-canonical model of X is divisorial, and thus Y has worse than terminal singularities. This will contradict the assumption that $X \to \mathrm{Spec}(\mathbb{C})$ is a rank 2 fibration, as it violates condition (3) in Definition 4.1.5.

First note that the structure sequence of $C \subset \mathbb{P}^3$ induces the following inequality:

$$h^0(\mathbb{P}^3, \mathcal{I}_C(3)) \geq h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)) - h^0(C, \mathcal{O}_C(3H)) \geq 10 - h^0(\hat{C}, \mathcal{O}_{\hat{C}}(3H)), \tag{4.3}$$

where $\nu \colon \hat{C} \to C$ is the normalization of C and the second inequality stems from the injectivity of ν^* on global sections. Using the Riemann-Roch theorem on $\hat{C}$, we conclude that $h^0(\mathbb{P}^3, \mathcal{I}_C(3)) \geq 1$, i.e., C is contained in a cubic surface T. Moreover, $S \cap T = C \cup \ell$ and this cubic surface is unique for degree reasons. By analyzing the divisor $5H - C = 2H + \ell$ on S, we see that there is a quintic surface containing C whose strict transform on X does not meet $\Gamma \subset X$. Hence, the ideal of C in $\mathbb{P}^3$ is $I_C = (f_3, f_4, f_5)$, where f_i are homogeneous polynomials of degree i, with f_3 and f_4 cutting out T and S, respectively. The anti-canonical rational contraction of X factors through the rational map of $\mathbb{P}^3$ given by the global sections of $\mathcal{I}_C(4)$. We may choose coordinates and a basis of $H^0(\mathbb{P}^3, \mathcal{I}_C(4))$ so that this map $\mathbb{P}^3 \dashrightarrow \mathbb{P}^4$ is given by $(x_0 f_3 : x_1 f_3 : x_2 f_3 : x_3 f_3 : f_4)$, and its image is the singular hypersurface $Y \subset \mathbb{P}^4$ cut out by the equation

$$f_4(y_0, y_1, y_2, y_3) - y_4 f_3(y_0, y_1, y_2, y_3) = 0.$$

This map is divisorial, contracting the strict transform of T in X to the singular point $(0 : 0 : 0 : 0 : 1) \in Y$. Furthermore, since the map is anti-canonical, Y has strictly canonical singularities, contradicting the assumption that $X \to \mathrm{Spec}(\mathbb{C})$ is a rank 2 fibration. From this contradiction we conclude that X is weak Fano.

Finally, by Lemma 4.3.7, for a general $C' \in |C|$ the blowup $X' \to \mathbb{P}^3$ of $\mathbb{P}^3$ along C' is weak Fano and $X' \to \mathrm{Spec}(\mathbb{C})$ is a rank 2 fibration. Furthermore, by Proposition 3.5.2, we may choose C' to be smooth. Theorem 4.3.5 then implies that $(g(C'), \deg(C')) = (p_a, d)$ appears in (†). ☐

```
syms x y
% Preallocate memory for GD1 and GD2
GD = [];
GD1 = [];
GD2 = [];
for d = 1:15
    g_max = floor(d^2 / 8);
    for g = 0:g_max
        GD = [GD; g, d, d^2 - 8*(g-1)];
    end
end
t = size(GD, 1);

for i = 1:t
    g = GD(i, 1);
    d = GD(i, 2);
    discHC = GD(i, 3);
    Div = divisors(discHC);
    R = [];  %%% possible R=[r b c m n]

    for j = 1:length(Div)
        divHC = sqrt(discHC / Div(j));
        if mod(divHC, 1) == 0
            b = 4 - sqrt(mod(Div(j), 8));
            c = (b^2 - Div(j)) / 4;
            m1 = (d - b*divHC) / 4;
            m2 = (d + b*divHC) / 4;
            R = [R; Div(j), b, c, m1, divHC; Div(j), b, c, m2, -divHC];
        end
    end

    % Filter results where m is an integer
    R1 = R(mod(R(:, 4), 1) == 0, :);
    R2 = [];

    for l = 1:size(R1, 1)
        b = R1(l, 2);
        c = R1(l, 3);
        m = R1(l, 4);
        n = R1(l, 5);
```

```
        % Solve system of equations
        z = solve(4*x^2 + 2*b*x*y + c*y^2 + 2,
             (16-d)*x + (4*b - b*m - c*n)*y + 1, 'Real', true);
        XY = [double(z.x), double(z.y)];

        % Filter integer and real solutions
        for u = 1:size(XY, 1)
            if all(mod(XY(u, :), 1) ==0) && 4*XY(u, 1) + b*XY(u, 2) >0
                R2 = [R2; g, d, R1(l, :), XY(u, 1), XY(u, 2)];
            end
        end
    end
    GD1 = [GD1; R2];
end

% Filter final values
mod8 = mod(GD1(:, 3), 8);
valid_rows = GD1(:, 3) > 8 & (mod8 == 0 | mod8 == 1 | mod8 == 4);
GD2 = GD1(valid_rows, :);
```

Chapter 5

On Gizatullin's problem for low Picard rank

The goal of this chapter is to provide a complete solution to Gizatullin's problem (Problem 1) in the case of quartic surfaces $S \subset \mathbb{P}^3$ with Picard rank two. We divide the analysis into two cases: when the quartic surface S has a discriminant greater than 233, and when it has a discriminant less than or equal to 233. We prove Theorem A and Theorem B, which correspond precisely to these two cases, respectively. As a consequence of Theorem A, we obtain a negative answer to Oguiso's question (Problem 2). Throughout this chapter, we will adopt the following formulation of Gizatullin's problem.

Given a smooth quartic surface $S \subset \mathbb{P}^3$, consider the subgroup $\mathrm{Bir}(\mathbb{P}^3; S) \subset \mathrm{Bir}(\mathbb{P}^3)$ consisting of Cremona transformations that stabilize S, meaning that $\varphi \in \mathrm{Bir}(\mathbb{P}^3)$ and $\varphi_*(S) = S$. The restriction of an element $\varphi \in \mathrm{Bir}(\mathbb{P}^3; S)$ to S induces a birational self-map of S, which is an automorphism since S is a K3 surface. Therefore, we have a homomorphism of groups

$$\Psi\colon \mathrm{Bir}(\mathbb{P}^3; S) \to \mathrm{Aut}(S). \tag{5.1}$$

Hence, Problem 1 is equivalent to investigating the image of this map Ψ. More specifically, we aim to find conditions on S that determine whether this map is surjective or not.

An easy case to begin with is when the smooth quartic $S \subset \mathbb{P}^3$ has Picard rank $\rho(S) = 1$. First, we recall the following. Let H be a hyperplane section of a smooth quartic $S \subset \mathbb{P}^3$ (of any Picard rank). Since H is the intersection of S with an hyperplane $H_{\mathbb{P}^3}$ of $\mathbb{P}^3$, it is a very ample divisor on S with self-intersection $H^2 = (H_{\mathbb{P}^3})|_S \cdot (H_{\mathbb{P}^3})|_S = H_{\mathbb{P}^3}^2 \cdot S = 4$.

Proposition 5.0.1. Any smooth quartic $S \subset \mathbb{P}^3$ with Picard rank $\rho(S) = 1$ has trivial automorphism group $\mathrm{Aut}(S)$. Therefore, the map $\Psi\colon \mathrm{Bir}(\mathbb{P}^3; S) \to \mathrm{Aut}(S) = \{1\}$ is surjective.

Proof. Let H be a hyperplane section on S. Since $H^2 = 4$, H is a primitive element of $\operatorname{Pic}(S)$ and therefore generates the lattice. In fact, if $\operatorname{Pic}(S) = \mathbb{Z}W$ and $H = kW$ is a multiple of the generator with $k \neq \pm 1$, we have $4 = H^4 = k^2 W^2$. Since W^2 is an even number, it follows that k^2 must divide 2, which is a contradiction.

Therefore, we can write the Picard lattice as $\operatorname{Pic}(S) = \mathbb{Z}H = \langle 4 \rangle$. Let $f \in \operatorname{Aut}(S)$ be an automorphism of S. Note that the only classes on $\operatorname{Pic}(S)$ with self-intersection 4 are H and $-H$. Thus, $f^*H = H$ since the induced isometry f^* in $\operatorname{Pic}(S)$ preserves the ample cone, and therefore f acts trivially on $\operatorname{Pic}(S)$ and the discriminant group $A(\operatorname{Pic}(S))$.

Now, we observe that $f^* = \pm\,\mathrm{id}$ on the transcendental lattice $\mathrm{T}(S)$, by Proposition 3.4.10. In fact, $f^* = \mathrm{id}$, because otherwise the induced action on the discriminant group $A(\mathrm{T}(S))$ is $-\,\mathrm{id}$. This is not possible since $A(\mathrm{T}(S)) \cong A(\operatorname{Pic}(S)) \cong \mathbb{Z}_4$. Therefore, $f^* = \mathrm{id}$ on $H^2(S, \mathbb{Z})$ and hence f itself is trivial by Proposition 2.1.11 and Theorem 3.3.3. □

In higher Picard rank, the situation becomes more intriguing, as the automorphism group of smooth quartic surfaces (and K3 surfaces in general) is often infinite. In [Ogu12] and [Ogu13], Oguiso was the first to address Gizatullin's problem for higher Picard rank, constructing two examples of quartic surfaces that exhibit distinct behaviors. We present these examples below.

Example 5.0.2 ([Ogu12, Theorem 1.8] and [Ogu13, Theorem 1.2])**.** Let $b \geq 4$ and $S_b \subset \mathbb{P}^3$ be an Aut-general smooth quartic surface with $\rho(S_b) = 2$ and intersection matrix

$$\begin{pmatrix} 4 & 4b \\ 4b & 4 \end{pmatrix}.$$

Such quartics S_b exist and Oguiso showed that $\operatorname{Aut}(S_b) = \mathbb{Z}$ and $\operatorname{Bir}(\mathbb{P}^3; S_b) = \{1\}$. Thus, the image of the restriction map Ψ in (5.1) is trivial, which is equivalent to conclude that no non-trivial automorphism of S_b is induced by a Cremona transformation of $\mathbb{P}^3$.

The main argument used by Oguiso in this example is based on Takahashi's result (Proposition 4.1.4). For the case of $Y = \mathbb{P}^3$, $\operatorname{Pic}(\mathbb{P}^3)$ has rank one and it is generated by the class of hyperplanes, and the index is $s = 4$. Thus, Proposition 4.1.4 asserts that if there exists a non-regular Cremona transformation that preserves a quartic surface $S \subset \mathbb{P}^3$, it must lead to the existence of a curve $C \subset S$ with degree $d < 16$, which is not a complete intersection of S with another surface in $\mathbb{P}^3$. The surfaces S_b considered here do not satisfy this condition. In the next section, we build on this approach to explore Gizatullin's Problem for more general smooth quartic surfaces $S \subset \mathbb{P}^3$ with $\rho(S) = 2$.

The following is the other example constructed by Oguiso.

Example 5.0.3 ([Ogu12, Theorem 1.7])**.** Consider a smooth quartic surface $S \subset \mathbb{P}^3$ with $\rho(S) = 3$

and intersection matrix

$$\begin{pmatrix} 4 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{pmatrix}.$$

Oguiso showed that $\mathrm{Aut}(S) = \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ and the restriction map $\Psi\colon \mathrm{Bir}(\mathbb{P}^3; S) \to \mathrm{Aut}(S)$ is surjective. Therefore, every automorphism of S is induced by a Cremona transformation of $\mathbb{P}^3$.

5.1 The high discriminant case

In this subsection we provide the first steps towards a complete answer to Gizatullin's problem when $\rho(S) = 2$. This is the subject of Theorem A, which recovers Oguiso's example 5.0.2. We start by applying results in Sections 2.1.1 and 3.5 to the specific case of smooth quartic surfaces with Picard rank $\rho(S) = 2$.

Let $S \subset \mathbb{P}^3$ be a smooth quartic surface with Picard rank $\rho(S) = 2$ and H be a hyperplane section. Recall that H is a very ample divisor on S with $H^2 = 4$. As a consequence of Lemma 2.1.17, the Picard lattice can be always expressed as $\mathrm{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}W$ for some divisor class W. Thus, with respect to this basis, the intersection product is given by the following matrix

$$Q = \begin{pmatrix} 4 & b \\ b & 2c \end{pmatrix}, \tag{5.2}$$

for some $b, c \in \mathbb{Z}$. The discriminant of S, $r(S) = -\operatorname{disc}(\mathrm{Pic}(S)) = b^2 - 8c$, is a positive integer congruent to $0, 1, 4$ module 8. This follows because any square number is congruent to $0, 1, 4$ module 8.

Lemma 5.1.1. Let $S \subset \mathbb{P}^3$ be a smooth quartic surface with $\rho(S) = 2$ and H a hyperplane section. Then, $r(S) > 8$ and the Picard lattice of S is determined by the discriminant $r(S)$, in the sense of Proposition 2.1.18. In particular, if b', c' are integers such that $(b')^2 - 8c' = r(S)$, then there exists a divisor $D \in \mathrm{Pic}(S)$ such that $D \cdot H = b'$, $D^2 = 2c'$ and $\mathrm{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}D$.

Proof. Write $\mathrm{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}W$ such that the intersection product is given by the matrix Q in (5.2). To prove that $\mathrm{Pic}(S)$ is determined by $r(S)$ we see that 4 satisfies the condition (2.5) of Proposition 2.1.18. Indeed, for any integer n, its squared $n^2 \equiv 0, 1, 4 \mod 8$. The three cases are the following. (i) $n^2 \equiv 0 \mod 8$ if and only if $n \equiv 0 \mod 4$, (ii) $n^2 \equiv 1 \mod 8$ if and only if $n \equiv 1, 3 \mod 4$, and (iii) $n^2 \equiv 4 \mod 8$ if and only if $n \equiv 2 \mod 4$; which clearly implies that the condition follows.

Now, let $b', c' \in \mathbb{Z}$ such that $(b')^2 - 8c' = r(S)$. By condition (2.5) of Proposition 2.1.18 we have that

$b - b' \equiv 0 \mod 4$ or $b + b' \equiv 0 \mod 4$. Thus,

$$D = \frac{b' - b}{4} H + W \quad \text{or} \quad D = \frac{b' + b}{4} H - W$$

is an element of $\mathrm{Pic}(S)$ satisfying that $D \cdot H = b'$, $D^2 = 2c'$. Moreover, since D is not a multiple of H, the sublattice $\langle H, D \rangle$ of $\mathrm{Pic}(S)$ has rank two and same discriminant. Thus, $\mathrm{Pic}(S)$ is generated by H and D, by Proposition 2.1.1(3).

Finally, we exclude the cases when $r(S) = 1, 4$ ou 8. By the reasoning above, for $r(S) = 1, 4, 8$, there exists a divisor D such that Picard lattice $\mathrm{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}D$ and the intersection matrix, with respect to this basis, is

$$\begin{pmatrix} 4 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 4 & 2 \\ 2 & 0 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}.$$

When $r(S) = 1$, D is a divisor with $D \cdot H = 1$ and $D^2 = 0$. Write $D = M + F$, where F is the fixed part of the linear system $|D|$. Since $M \neq 0$ and $M \cdot H \geq 1$, from $1 = D \cdot H = M \cdot H + F \cdot H$ we obtain that $H \cdot M = 1$ and $D = M$. Thus, D is an irreducible curve, by Proposition 3.1.8(4). This contradicts that H is very ample.

When $r(S) = 4$, D is a divisor with $D \cdot H = 2$ and $D^2 = 0$. Write $D = M + F$, where F is the fixed part of the linear system $|D|$. Note that $M \cdot H \geq 1$, and $F \cdot H \geq 0$ with equality if and only if $F = 0$. Assume $F \neq 0$. Then $F \cdot H = 1$ and $M \cdot H = 1$. In this case, F is a rational curve by Proposition 3.1.8(2). Representing $F = nH + mD$ and looking for integer solutions of the equation $-2 = F^2 = 4n^2 + 2nm$, we get that $F = H - 3D$. Thus, from $1 = F \cdot D$ and $0 = D^2 = D \cdot (M + F) = D \cdot M + 1$ we get that $D \cdot M = -1$, which contradicts Proposition 3.1.8(1). So, $F = 0$, $D = M$ and $D = aE$, where E is an irreducible curve and $a \in \{1, 2\}$, by Proposition 3.1.8(4). Thus, $E \cdot H \in \{1, 2\}$ and again contradicts that H is very ample, by Proposition 3.1.7.

When $r(S) = 8$, D is a divisor with $D \cdot H = 0$ and $D^2 = -2$. Proposition 3.1.5(1) implies that D or $-D$ is linearly equivalent to a effective divisor. Thus, we get that $D \cdot H > 0$ or $-D \cdot H > 0$: a contradiction. □

By Theorem 3.4.8, given an even lattice of signature $(1, 1)$, it occurs as the Picard Lattice of some K3 surface. In the following result, we add conditions on the Picard lattice so that this K3 surface is in fact a smooth quartic in $\mathbb{P}^3$. We say that a lattice L represents an even integer $2k$ if there is an element $x \in L$ with $x^2 = 2k$.

Proposition 5.1.2. Let S be a K3 surface with $\rho(S) = 2$. Assume that the Picard lattice $\mathrm{Pic}(S)$ represents 4 and the discriminant $r(S) > 8$. Then, there is a very ample divisor $H \in \mathrm{Pic}(S)$, such that $H^2 = 4$ and the induced embedding $\iota_{|H|} \colon S \hookrightarrow \mathbb{P}^3$ is an isomorphism onto a smooth quartic surface.

Proof. Let S be as the hypothesis and $H \in \mathrm{Pic}(S)$ with $H^2 = 4$. We want to apply Proposition 3.1.7 to conclude that H is a very ample line bundle and hence it induces and embedding of S as a smooth quartic in $\mathbb{P}^3$. For that, first we have to prove that H is a nef divisor. We consider $\Delta = \{b \in \mathrm{Pic}(S) \mid b^2 = -2 \text{ and } b \text{ is effective}\}$, given $b \in \Delta$, we can define a Picard-Lefchetz reflection s_b as

$$\begin{aligned} s_b \colon Pic(S) &\to Pic(S) \\ D &\mapsto D + (b \cdot D)b \end{aligned}$$

By [BPVdV84, chapter VIII, proposition 3.9], the closure $\overline{\mathrm{Amp}(S)} \cap \mathcal{C}^+$ is a strict fundamental domain for the action of the group of Picard-Lefchetz reflections on the positive cone $\mathcal{C}^+$. This means that, given $s_b(H)$, there is a nef divisor D' and $b' \in \Delta$, such that $s_b(H) = s'_b(D')$, since the Picard-Lefchetz reflections are isometries, we have $H^2 = D'^2 = 4$ and, after replacing H with D' if necessary, we can suppose that H is nef.

Since H is a primitive element of the lattice, $\mathrm{Pic}(S) = \langle H, W \rangle$, for some element W, and the intersection matrix is given by Q in (5.2). Note that the second condition of Proposition 3.1.7 is immediately satisfied. Suppose that there is an irreducible curve E with $(E^2, H \cdot E) \in \{(0,1), (0,2), (-2,0)\}$. Consider the sublattice $\langle H, E \rangle$ of $Pic(S)$. This is a rank two lattice with discriminant $4E^2 - (H \cdot E)^2 = -1, -4, -8$, respectively. By Proposition 2.1.1(3), $r(S)$ divides $1, 4$ and 8 respectively. We arrive at a contradiction since $r(S) > 8$. Therefore, we conclude that all conditions of Proposition 3.1.7 are satisfied and H is very ample. Finally, the image of $\iota_{|H|}$ is a smooth quartic of $\mathbb{P}^3$ since $H^2 = 4$. □

Remark 5.1.3. For sake of completeness, we describe the situation of K3 surfaces with $\rho(S) = 2$, $r(S) = 8$ and $\mathrm{Pic}(S)$ representing 4. Let H be a nef divisor with $H^2 = 4$. Since H is a primitive element and as in Lemma 5.1.1, there is exits a divisor D such that $\mathrm{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}D$ and the intersection matrix has the form

$$\begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}.$$

Since $r(S)$ is not a square number and D has self-intersection -2, the Mori cone $\overline{\mathrm{NE}}(S)$ s generated by two rational curves, by Proposition 3.5.1. One can prove that D is one of the two rational curves. Moreover, the rational map $\varphi_{|H|} \colon S \to \mathbb{P}^3$ induced by $|H|$ is a morphism contracting uniquely the rational curve D. Therefore, the image $\tilde{S}$ of S under this morphism is a quartic in $\mathbb{P}^3$ is smooth apart from a unique A_1-singularity and $\mathrm{Cl}(\tilde{S}) \cong \mathbb{Z}$. Gizatullin's problem for $\tilde{S} \subset \mathbb{P}^3$ was solved in [ACM23, Theorem B]. They conclude that for a general such $\tilde{S} \in \mathbb{P}^3$, any automorphism comes from a Cremona transformation of the ambient space.

Now, we investigate the automorphism group $\mathrm{Aut}(S)$ of S. We first rule out some possibilities for the Picard lattice.

Proposition 5.1.4. The lattices $U(k)$ for $k \in \{1, 2, 3, 11\}$, $\langle 2\rangle \oplus \langle -2\rangle$ and H_5 do not occur as the Picard lattice of a smooth quartic surface $S \subset \mathbb{P}^3$ with $\rho(S) = 2$.

Proof. By Lemma 5.1.1 and Proposition 5.1.2, a hyperbolic lattice L of rank two occurs as the lattice of a smooth quartic surface $S \subset \mathbb{P}^3$ if and only if L represents 4 and $\mathrm{disc}(L) > 8$. The lattices U, $U(2)$, $\langle 2\rangle \oplus \langle -2\rangle$ and H_5 have discriminant $-1, -4, -4, -5$, respectively. Moreover, the lattices $U(3)$ and $U(11)$ do not represent 4, since the self-intersection of any element x is a multiple of 3 or 11. Hence, we have proven the desired result. □

Proposition 5.1.5. Let $S \subset \mathbb{P}^3$ be a smooth quartic with $\rho(S) = 2$. Any non-trivial automorphism f of S of finite order has order two with invariant lattice $H^2(S, \mathbb{Z})^{f^*} = \langle A\rangle$, for a unique ample class $A \in \mathrm{Pic}(S)$ with $A^2 = 2$.

Proof. Let $f \in \mathrm{Aut}(S)$ be a non-trivial automorphism of finite order n and p a prime factor of n. The automorphism $g := f^{n/p}$ has order p, it is non-symplectic and $0 \neq H^2(S, \mathbb{Z})^{g^*} \subset \mathrm{Pic}(S)$ by Proposition 3.5.3. Thus, $H^2(S, \mathbb{Z})^{g^*}$ has either rank one or two. In the latter case, $\mathrm{Pic}(S) = H^2(S, \mathbb{Z})^{g^*}$ is one of the following possibilities: U, $U(2)$, $U(3)$, $U(11)$, $\langle 2\rangle \oplus \langle -2\rangle$ or H_5, by Proposition 3.5.3. This contradicts Proposition 5.1.4. Hence $p = 2$ and we are in case (2) of Proposition 3.5.5, i.e., $H^2(S, \mathbb{Z})^{g^*} = \langle A\rangle$, for some ample class $A \in \mathrm{Pic}(S)$ with $A^2 = 2$. In particular, this implies that f is an automorphism of order $n = 2^k$. Our goal now is to prove that $k = 1$, thus $f = g$ and we obtain the desired conclusion.

Assume $k \geq 2$, thus $h = f^{n/4}$ is an automorphism of order 4 and the automorphism g of order two above satisfies $g = f^{n/2} = h^2$. Moreover, $H^2(S, \mathbb{Z})^{h^*}$ is not trivial. The induced isometry $h^* \neq \mathrm{id}$ of $H^2(S, \mathbb{Z})$ has also order 4 and it could be $h^* = \mathrm{id}$, or has order 2 or 4 on the Picard lattice $\mathrm{Pic}(S)$, which is not possible. Indeed, it does not have order 4 by Lemma 2.1.20, and if $h^* = \mathrm{id}$ or it has order 2, we get that $g^* = \mathrm{id}$ on $\mathrm{Pic}(S)$ and so, $H^2(S, \mathbb{Z})^{g^*} = \mathrm{Pic}(S)$; which is not possible. □

Remark 5.1.6. The proof of Proposition 3.5.5(2) establishes that, in particular, every involution of a smooth quartic surface $S \subset \mathbb{P}^3$ with Picard rank $\rho(S) = 2$ is geometrically realizable as the cover involution of the double cover $S \longrightarrow \mathbb{P}^2$ associated to a unique ample class $A \in \mathrm{Pic}(S)$ with $A^2 = 2$.

Recall the group $\mathrm{Aut}(\mathbb{P}^3; S) = \{\varphi \in \mathrm{Aut}(\mathbb{P}^3) | \varphi(S) = S\}$ of the regular maps of $\mathbb{P}^3$ stabilizing S. Matsumura and Monsky proved in [MM64, Theorem 1] that any regular map of $\mathbb{P}^3$ stabilizing a smooth quartic surface $S \subset \mathbb{P}^3$ (of any Picard rank) induces a finite order automorphism of S. A proof of this fact can also be found in [Ogu13, Theorem 3.2], which we use to obtain the following result.

Proposition 5.1.7. Let $S \subset \mathbb{P}^3$ be a smooth quartic surface with $\rho(S) = 2$. Then $\mathrm{Aut}(\mathbb{P}^3; S) = \{1\}$, i.e., no non-trivial automorphism of S comes from automorphism of $\mathbb{P}^3$.

Proof. The map $\mathrm{Aut}(\mathbb{P}^3; S) \to \mathrm{Aut}(S)$ defined by the restriction $\varphi \mapsto \varphi|_S$ is well-defined and injective. By Section 3.3, $\mathrm{Aut}(S)$ is discrete and so is $\mathrm{Aut}(\mathbb{P}^3; S)$. Moreover, $\mathrm{Aut}(\mathbb{P}^3) = PGL_4(\mathbb{C})$ is an affine variety and $\mathrm{Aut}(\mathbb{P}^3; S)$ is a Zariski closed, which then is finite.

Let $\varphi \in \mathrm{Aut}(\mathbb{P}^3; S)$ be a non-trivial regular map of $\mathbb{P}^3$ and set $f := \varphi|_S \in \mathrm{Aut}(S)$. Then f is not trivial and has order two, by the reasoning above and Proposition 5.1.5. Since φ is an automorphism of $\mathbb{P}^3$ stabilizing S, f preserves the hyperplane class H, i.e., $f^*H = H$. Thus, $H \in H^2(S, \mathbb{Z})^f$ must be a multiple of an ample divisor A with $A^2 = 2$, which is absurd since H is primitive. □

The last proposition tells us that it remains to investigate non-regular Cremona transformations of $\mathbb{P}^3$ preserving the quartic S. Proposition 4.1.4 asserts that the existence of a non-regular Cremona transformation stabilizing a quartic surface $S \subset \mathbb{P}^3$ forces the existence of a curve $C \subset S$ of degree $d < 16$ that is not a complete intersection of S with another surface in $\mathbb{P}^3$. Combining this with Proposition 5.1.7, we conclude that Problem 1 is solved for surfaces that do not contain curves of degree < 16 that are not complete intersection. This is a condition that can be read off from the Picard lattice of the quartic, which is determined by its discriminant. Indeed, the main result of this section is the following.

Theorem 5.1.8 (Theorem A)**.** Let $S \subset \mathbb{P}^3$ be a smooth quartic surface with $\rho(S) = 2$ and discriminant $r(S) > 233$. Then, there does not exist a curve $C \subset S$ of degree < 16 that is not a complete intersection. Consequently, $\mathrm{Bir}(\mathbb{P}^3; S) = \{1\}$.

Proof. Let $S \subset \mathbb{P}^3$ be as the hypothesis and H the class of a hyperplane section. Thus, we can write $\mathrm{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}W$ for some divisor W, and the intersection matrix is given by Q as in (5.2). We will prove that every curve $C \subset S$ that has degree $d < 16$ is of the form $C = S \cap T$, where T is a hypersurface of $\mathbb{P}^3$. Hence the result follows from Propositions 4.1.4 and 5.1.7.

Given a curve $C \subset S$, it has degree $d = H \cdot C$ and self-intersection $C^2 \geq -2$ by Proposition 3.1.5(2). Assume $d < 16$ and $C = mH + nW$ for some integers m, n. From Lemma 2.1.16(1), we get that

$$-8 \leq 4C^2 = d^2 - r(S)n^2 \leq 225 - r(S)n^2 \implies 233n^2 < r(S)n^2 \leq 233.$$

As n is an integer number, the last inequality is satisfied only if $n = 0$. Therefore, $C = mH$ in $\mathrm{Pic}(S)$.

Now, denote by $\iota = \varphi_{|H|} \colon S \hookrightarrow \mathbb{P}^3$ the corresponding embedding. The following exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-4+m) \longrightarrow \mathcal{O}_{\mathbb{P}^3}(m) \longrightarrow \iota_*\mathcal{O}_S(m) \longrightarrow 0$$

induces the following one in cohomology

$$H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(m)) \longrightarrow H^0(\mathbb{P}^3, \iota_*\mathcal{O}_S(m)) \longrightarrow H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-4+m)).$$

Since $H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(l)) = 0$ for any $l \in \mathbb{Z}$, the first map in the last exact sequence is surjective, which implies that C is the intersection of S and a hypersurface T of $\mathbb{P}^3$. □

We point out that Theorem 5.1.8 generalizes Example 5.0.2. Indeed, the surfaces S_b in Example 5.0.2 are assumed to be Aut-general, as symplectic and anti-symplectic automorphisms are easier to handle. This assumption ensures that $\mathrm{Aut}(S_b) = \mathbb{Z}$ and allows Oguiso to conclude that no non-trivial automorphism of S_b arises from $\mathrm{Bir}(\mathbb{P}^3)$. However, if they are not Aut-general, their automorphism group contains $\mathbb{Z}$ as a finite index subgroup, and the realization of the whole automorphism group $\mathrm{Aut}(S)$ as elements of $\mathrm{Bir}(\mathbb{P}^3)$ still holds since $r(S_b) = 16(b^2 - 1) > 233$.

5.1.1 A counter-example for Oguiso's question

In this subsection, we present smooth quartic surfaces where no non-trivial automorphism of the surfaces arises from any Cremona transformation. Specifically, these surfaces possess finite-order automorphisms, providing a negative answer to Problem 2 posed by Oguiso.

Example 5.1.9. Let $b \geq 16$ be an integer and $S = S_b \subset \mathbb{P}^3$ be a smooth quartic surface with Picard lattice $\mathrm{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}A$ and intersection matrix given by

$$\begin{pmatrix} 4 & b \\ b & 2 \end{pmatrix}.$$

Since $r(S) = b^2 - 8 > 233$, no non-trivial elements of $\mathrm{Aut}(S)$ are induced by birational maps in $\mathrm{Bir}(\mathbb{P}^3)$. Moreover, $\mathrm{Aut}(S)$ contains automorphisms of finite order since at least a copy of $\mathbb{Z}_2$ is inside it. Recall by Proposition 5.1.5 that any finite order automorphism is an involution. Therefore, the involution that generates $\mathbb{Z}_2$, and hence all involutions of S, are not the restriction of a Cremona transformation of $\mathbb{P}^3$. Let us briefly show the existence of these surfaces. Notice that any rank two lattice with bilinear form given by the matrix above is even with signature $(1, 1)$. By Theorem 3.4.8, there exists a K3 surface S with $\mathrm{Pic}(S)$ as above, and by Proposition 5.1.2, every such K3 S is a smooth quartic in $\mathbb{P}^3$. Moreover, A is an ample divisor with $A^2 = 2$, by Proposition 3.5.2. Hence, the finite index subgroup $\mathrm{Aut}^{\pm}(S) \subset \mathrm{Aut}(S)$ of symplectic and anti-symplectic automorphisms of S is either $\mathrm{Aut}^{\pm}(S) = \mathbb{Z}_2$ or $\mathrm{Aut}^{\pm}(S) = \mathbb{Z}_2 * \mathbb{Z}_2$ by Corollary 3.5.6.

5.2 The low discriminant case

In this section, we examine the remaining cases to provide a complete resolution of Gizatullin's problem. Let $S \subset \mathbb{P}^3$ be a smooth quartic surface with Picard rank 2, and $\varphi\colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ a birational map. As explained in Remark 4.2.8, the condition $\varphi(S) = S$ is satisfied if and only if φ is volume preserving with respect to the Calabi-Yau pair $(\mathbb{P}^3, S)$. Therefore, we employ the volume-preserving Sarkisov program introduced in Section 4.2.

According to Takahashi's result, the classical Sarkisov program establishes that the non-equiality $\mathrm{Bir}(\mathbb{P}^3; S) \neq \mathrm{Aut}(\mathbb{P}^3; S)$ imposes constraints on the degrees of curves on S that are not complete intersections. By employing the specialized volume-preserving version of the Sarkisov program, we additionally derive constraints on the genus of such curves. This leads us to the following proposition.

Proposition 5.2.1. Let $S \subset \mathbb{P}^3$ be a smooth quartic surface with Picard rank 2. Suppose that $\mathrm{Bir}(\mathbb{P}^3; S) \neq \mathrm{Aut}(\mathbb{P}^3; S)$. Then there is a smooth curve $C \subset S$ of genus g and degree d such that the pair (g, d) belongs to the list (†) of Theorem 4.3.5.

Proof. Suppose that there exists a birational map $\varphi \in \mathrm{Bir}(\mathbb{P}^3; S) \setminus \mathrm{Aut}(\mathbb{P}^3; S)$. By Theorem 4.2.11, there exists a factorization of φ as a composition of volume preserving Sarkisov links. Since $\rho(\mathbb{P}^3) = 1$, the first Sarkisov link in the decomposition necessarily starts with a volume preserving divisorial contraction $X \to \mathbb{P}^3$. By Proposition 4.2.12, $X \to \mathbb{P}^3$ is the blowup of $\mathbb{P}^3$ along a curve $C' \subset S$. By Lemma 4.3.2, $X \to \mathrm{Spec}(\mathbb{C})$ is a rank 2 fibration, and so Proposition 4.3.11 implies that $(p_a(C'), \deg(C'))$ belongs to the list (†). A general member C in the linear system $|C'|$ of S is a smooth curve of genus g and degree d with $(g, d) = (p_a(C'), \deg(C'))$, and the result follows. □

In what follows, we adopt the notation introduced in Section 5.1. In particular, we denote by H the class of a hyperplane section of $S \subset \mathbb{P}^3$. Given a curve $C \subset S$, the arithmetic genus p_a and degree d of C are given by $(p_a, d) = \left(\frac{C^2}{2} + 1, C \cdot H\right)$.

Corollary 5.2.2. Let S be a smooth quartic surface with Picard rank 2 and discriminant $r(S)$. If $r(S) > 57$ or $r(S) = 52$, then

$$\mathrm{Bir}(\mathbb{P}^3; S) = \mathrm{Aut}(\mathbb{P}^3; S) = \{1\}.$$

Proof. Let S be a smooth quartic surface with Picard rank 2 and discriminant $r(S)$. Recall from Section 5.1 that $r(S) \equiv 0, 1, 4 \pmod 8$. Suppose that there exists a birational map $\varphi \in \mathrm{Bir}(\mathbb{P}^3; S) \setminus \mathrm{Aut}(\mathbb{P}^3; S)$. By Proposition 5.2.1, there is a curve $C \subset S$ with arithmetic genus p_a and degree d satisfying $(p_a, d) = \left(\frac{C^2}{2} + 1, C \cdot H\right) \in$ (†). Consider the sublattice L of $\mathrm{Pic}(S)$ spanned by H and C. It has rank 2 and the opposite of its discriminant $r' = -\mathrm{disc}(L) = (C \cdot H)^2 - 4C^2 = d^2 - 8(p_a - 1)$. We compute r' for each pair $(p_a, d) \in$ (†), and list the possible values of r' in the same order as the corresponding pair in (†):

$$r' \in \left\{ \begin{array}{c} 9,\ 12, 17, 24, 33, 44, 57,\ 9,\ 16, 25, 36, 49, \\ 17, 28, 41, 56, 20, 33, 48, 12, 25, 40, 17, 32, \\ 24,\ 41,\ 16,\ 33,\ 25,\ 17,\ 9,\ 28,\ 20,\ 17 \end{array} \right\}.$$

Since $r(S) = -\operatorname{disc}\big(\operatorname{Pic}(S)\big)$ must divide $r' = -\operatorname{disc}(L) \leq 57$, we conclude that $r(S) \leq 57$. Among the integers $r(S)$ with $r(S) \equiv 0, 1, 4 \pmod 8$ and $9 \leq r(S) \leq 57$, the only one that does not divide any r' in the above list is $r(S) = 52$. □

For quartic surfaces with Picard rank 2, Corollary 5.2.2 reduces Problem 1 to surfaces S with discriminant $r(S) \leq 57$ and $r(S) \neq 52$. The next proposition describes the automorphism group $\operatorname{Aut}(S)$ in these cases.

Proposition 5.2.3. The sets

$\mathcal{R}_0 = \{9, 12, 16, 24, 25, 33, 36, 44, 49, 57\}$,

$\mathcal{R}_1 = \{17, 41\}$,

$\mathcal{R}_2 = \{28, 56\}$, and

$\mathcal{R}_3 = \{20, 32, 40, 48\}$

give a partition of all integers $r(S) \leq 57$, $r(S) \neq 52$, such that $r(S)$ is the discriminant of a smooth quartic surface $S \subset \mathbb{P}^3$ with $\rho(S) = 2$.

If the quartic surface S is Aut-general, then its automorphism group is described as follows:

$$\operatorname{Aut}(S) \cong \begin{cases} \{1\}, & \text{if } r \in \mathcal{R}_0; \\ \mathbb{Z}_2, & \text{if } r \in \mathcal{R}_1; \\ \mathbb{Z}_2 * \mathbb{Z}_2, & \text{if } r \in \mathcal{R}_2; \\ \mathbb{Z}, & \text{if } r \in \mathcal{R}_3. \end{cases}$$

Proof. As before, we denote by H the class of a hyperplane section of S and let $\{H, W\}$ be a basis of $\operatorname{Pic}(S)$. With respect to this basis, the intersection product in $\operatorname{Pic}(S)$ is given by the matrix (5.2), and $r(S) \equiv 0, 1, 4 \pmod 8$. By Lemma 5.1.1, $r(S) > 8$. By Theorem 3.4.8, and Proposition 5.1.2, every even lattice with bilinear form given by (5.2), signature $(1, 1)$ and discriminant $8 < r(S) \leq 57$ can be realized as the Picard lattice of a smooth quartic surface. This proves the first assertion.

Suppose now that S is Aut-general, i.e., $\mathrm{Aut}(S) = \mathrm{Aut}^{\pm}(S)$. By Proposition 3.5.6, the automorphism group of S is completely determined by the existence of a divisor D with $D^2 \in \{0, -2\}$ and an ample divisor A with $A^2 = 2$. By Lemma 2.1.16, the existence of a divisor Δ on S with $\Delta^2 = k$, can be determined by the existence of an integer solution of the generalized Pell equation $x^2 - r(S)y^2 = 4k$. The second assertion then follows from checking the existence of integer solutions of the corresponding generalized Pell equations for each value of $r(S) \in \mathcal{R}_i$, $i \in \{0, 1, 2, 3\}$.

For each $r(S) \in \mathcal{R}_0 \cup \mathcal{R}_1$, either the equation $x^2 - r(S)y^2 = 0$ or the equation $x^2 - r(S)y^2 = -8$ has an integer solution, as illustrated in the following table. This implies that $\mathrm{Aut}(S) = \{1\}$ or $\mathrm{Aut}(S) \cong \mathbb{Z}_2$.

$r(S)$	9	12	16	17	24	25	33	36	41	44	49	57
(x, y)	$(1, 1)$	$(2, 1)$	$(4, 1)$	$(3, 1)$	$(4, 1)$	$(5, 1)$	$(5, 1)$	$(6, 1)$	$(19, 3)$	$(6, 1)$	$(7, 1)$	$(7, 1)$
$x^2 - r(S)y^2$	-8	-8	0	-8	-8	0	-8	0	-8	-8	0	-8

Suppose that $r(S) \in \mathcal{R}_0 = \{9, 12, 16, 24, 25, 33, 36, 44, 49, 57\}$. In order to show that $\mathrm{Aut}(S) = \{1\}$, we will show that there are no divisors with square 2, or equivalently that $x^2 - r(S)y^2 = 8$ does not have integer solutions. If $r(S) \in \{9, 12, 24, 33, 36, 44, 57\}$, then either $r(S) \equiv 0 \pmod 3$ or $r(S) \equiv 0 \pmod{11}$. So the equation $x^2 - r(S)y^2 = 8$ reduces to either $x^2 \equiv 2 \pmod 3$ or $x^2 \equiv 8 \pmod{11}$, and one checks easily that these have no integer solution. If $r(S) \in \{16, 25, 49\}$, then $r(S) = t^2$ for an appropriate integer $t > 1$. We set $z = ty$ and rewrite the equation $x^2 - r(S)y^2 = 8$ as $x^2 - z^2 = 8$. An integer solution (x, z) must satisfy $x^2 > z^2 > 1$. Then, from

$$8 = x^2 - z^2 = |x|^2 - |z|^2 \geq |x|^2 - (|x| - 1)^2 = 2|x| - 1,$$

we conclude that $2 \leq |z| < |x| \leq 4$, and one checks easily that there are no integer solutions.

Suppose that $r(S) \in \mathcal{R}_1 = \{17, 41\}$. If $r(S) = b^2 - 8c = 17$, then $(x, y) = (5, 1)$ and $(5, -1)$ are solutions of $x^2 - 17y^2 = 8$ and one of them satisfies that $z := \frac{x - yb}{4} \in \mathbb{Z}$. For such pair (x, y), $A = zH + yW \in \mathrm{Pic}(S)$ is the corresponding divisor on S with $A^2 = 2$. Note that $A \cdot H = 5$, $\mathrm{Pic}(S) = \langle H, A \rangle$, and A is effective. By Proposition 3.5.2, A is nef (and big). To show that A is ample, it is enough to check that there is no rational curve Γ such that $A \cdot \Gamma = 0$. Indeed, if there is such a curve Γ, then $E = A + \Gamma \in \mathrm{Pic}(S)$ satisfies $E^2 = 0$, and so $0 = 4E^2 = (H \cdot E)^2 - 17m^2$, where $m \in \mathbb{Z}$ is such that $E = nH + mW$ in $\mathrm{Pic}(S)$. This is not possible since $r(S) = 17$ is not a square number. When $r(S) = 41$, we argue in the same way, with $(x, y) = (7, 1), (7, -1)$ being solutions of $x^2 - 41y^2 = 8$.

If $r(S) \in \mathcal{R}_2 = \{28, 56\}$, then $r(S) \equiv 0 \pmod 7$. So the equation $x^2 - r(S)y^2 = -8$ reduces to $x^2 \equiv 6 \pmod 7$, which does not have solutions. Together with the fact that $r(S)$ is not a square number, this implies the non-existence of divisors on S with self-intersection 0 or -2. In order to show that $\mathrm{Aut}(S) \cong \mathbb{Z}_2 * \mathbb{Z}_2$, we must verify the existence of $A \in \mathrm{Pic}(S)$ ample with $A^2 = 2$. Indeed, the pair $(x,y) = (6,1)$ (respectively $(x,y) = (8,1)$) is a solution of the equation $x^2 - r(S)y^2 = 8$ for $r(S) = 28$ (respectively $r(S) = 56$), and the corresponding divisor A is automatically ample since S has no rational curves.

Finally, suppose that $r(S) \in \mathcal{R}_3 = \{20, 32, 40, 48\}$. Since $r(S)$ is not a square number, $x^2 - r(S)y^2 = 0$ does not have integer solutions. If $r(S) \in \{20, 40\}$, then $r(S) \equiv 0 \pmod 5$. So the equations $x^2 - r(S)y^2 = -8$ and $x^2 - r(S)y^2 = 8$ reduce to $x^2 \equiv 3$ and $x^2 \equiv 2 \pmod 5$, none of which has solutions. If $r(S) \in \{32, 48\}$, then write $r(S) = 16s$ for the appropriate integer $s \in \{2, 3\}$. If (x,y) is a solution of $x^2 - 16sy^2 = -8$ or $x^2 - 16sy^2 = 8$, then $x = 2z$ is an even integer. So these equations can be simplified to $z^2 - 4sy^2 = -2$ and $z^2 - 4sy^2 = 2$, and then reduced to $z^2 \equiv 2 \pmod 4$, which does not have solutions. $\square$

Next we show that whenever $r(S) \in \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$ we can find a curve $C \subset S$ as in Proposition 5.2.1 such that $\mathrm{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}C$, where H denotes the class of a hyperplane section. This allows us to describe explicit generators of $\mathrm{Aut}(S)$ via their action on $\mathrm{Pic}(S)$ in each case.

Proposition 5.2.4. Let S be an Aut-general smooth quartic surface with $\rho(S) = 2$ and discriminant $r(S) \in \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$.

1. There is a smooth curve $C \subset S$ of genus g and degree d such that $\mathrm{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}C$, where (g,d) depends on $r(S)$ as described in the following table:

$r(S)$	17	41	28	56	20	32	40	48
(g,d)	$(14,11)$	$(6,9)$	$(10,10)$	$(2,8)$	$(11,10)$	$(5,8)$	$(4,8)$	$(3,8)$
$\mathrm{Aut}(S)$	$\mathbb{Z}_2$		$\mathbb{Z}_2 * \mathbb{Z}_2$		$\mathbb{Z}$			

(C)

2. With respect to the basis $\{H, C\}$, the action of $\mathrm{Aut}(S)$ on $\mathrm{Pic}(S)$ is described as follows:

$r(S)$	17	28	20	40
$\mathrm{Aut}(S)$	$\left\langle \left(\begin{smallmatrix} 19 & 72 \\ -5 & -19 \end{smallmatrix}\right) \right\rangle$	$\left\langle \left(\begin{smallmatrix} 23 & 88 \\ -6 & -23 \end{smallmatrix}\right), \left(\begin{smallmatrix} -7 & -8 \\ 6 & 7 \end{smallmatrix}\right) \right\rangle$	$\left\langle \left(\begin{smallmatrix} 29 & 40 \\ -8 & -11 \end{smallmatrix}\right) \right\rangle$	$\left\langle \left(\begin{smallmatrix} 43 & 18 \\ -12 & -5 \end{smallmatrix}\right) \right\rangle$
$r(S)$	41	56	32	48
$\mathrm{Aut}(S)$	$\left\langle \left(\begin{smallmatrix} 27 & 104 \\ -7 & -27 \end{smallmatrix}\right) \right\rangle$	$\left\langle \left(\begin{smallmatrix} 31 & 120 \\ -8 & -31 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 & 0 \\ 8 & 1 \end{smallmatrix}\right) \right\rangle$	$\left\langle \left(\begin{smallmatrix} 41 & 24 \\ -12 & -7 \end{smallmatrix}\right) \right\rangle$	$\left\langle \left(\begin{smallmatrix} 209 & 56 \\ -56 & -15 \end{smallmatrix}\right) \right\rangle$

(M)

Proof. To prove (1), write $\mathrm{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}W$. As in (5.2), the intersection matrix of S with respect to the basis $\{H, W\}$ can be written as

$$Q = \begin{pmatrix} 4 & b \\ b & 2c \end{pmatrix},$$

so that $r(S) = b^2 - 8c$. Notice that for each $r(S) \in \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$, the corresponding pair (g, d) listed on table (C) satisfies $r(S) = d^2 - 8(g-1)$. By Lemma 5.1.1, there exists a divisor $D \in \mathrm{Pic}(S)$ satisfying $H \cdot D = d$ and $D^2 = 2g - 2$ and $\mathrm{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}D$. Furthermore, since $r(S) \neq 9$, Proposition 3.5.2 guarantees the existence of a smooth curve $C \in |D|$, allowing us to conclude that $\mathrm{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}C$.

To prove (2), we first notice that each matrix in the table represents an isometry $\phi \in O(\mathrm{Pic}(S))$ since $\phi^T Q \phi = Q$. Moreover, in each case, either $(\phi - \mathrm{id})Q^{-1} \in M_{2\times 2}(\mathbb{Z})$ or $(\phi + \mathrm{id})Q^{-1} \in M_{2\times 2}(\mathbb{Z})$. Therefore, by Proposition 3.5.7, the isometry $\phi \in O(\mathrm{Pic}(S))$ is induced by an automorphism $f \in \mathrm{Aut}(S)$ if and only if ϕH is ample. We now consider separately each case $r(S) \in \mathcal{R}_i$ for $i \in \{1, 2, 3\}$.

Suppose that $r(S) \in \mathcal{R}_3$. Then $\mathrm{Aut}(S) \cong \mathbb{Z}$ and S does not contain rational curves (see Corollary 3.5.6). We can check that $\phi H \cdot H > 0$ and $(\phi H)^2 > 0$, and so ϕH is ample by Proposition 3.1.6. Therefore, the isometry $\phi \in O(\mathrm{Pic}(S))$ is induced by an automorphism $f \in \mathrm{Aut}(S)$. For $r(S) = 20, 32, 40, 48$, the corresponding minimal integer solutions of $(*)$ are $(\alpha_1, \beta_1) = (4, 5), (7, 4), (43, 18)$ and $(4, 1)$, respectively. Moreover, $\phi = h^k$, where h is the matrix of Proposition 3.5.7(2) and $k = 3, 2, 1, 4$, respectively. In each case, ϕ is the minimal power of h satisfying the Gluing and Torelli conditions stated in Proposition 3.5.7, and so f is the generator of $\mathrm{Aut}(S)$.

Suppose that $r(S) \in \mathcal{R}_2$. Then $\mathrm{Aut}(S) \cong \mathbb{Z}_2 * \mathbb{Z}_2 \cong \mathbb{Z} \rtimes \mathbb{Z}_2$ and S does not contain rational curves (see Corollary 3.5.6). For a fixed $r(S) \in \mathcal{R}_2$, we denote by ϕ_1 and ϕ_2 the two isometries displayed in the table. Exactly as in the previous case, we check that $\phi_i H$ is ample, $i \in \{1, 2\}$, and so ϕ_1 and ϕ_2 are induced by automorphisms $f_1, f_2 \in \mathrm{Aut}(S)$ respectively. Notice that ϕ_1 and ϕ_2 have the form described in Proposition 3.5.7(1), and so f_1 and f_2 are involutions of S. To see that they are the generators of $\mathrm{Aut}(S)$, we check that $f_1 f_2$ generates the maximal copy of $\mathbb{Z}$ in $\mathrm{Aut}(S)$. Indeed, for $r(S) = 28, 56$, the minimal solutions of $(*)$ are $(\alpha_1, \beta_1) = (23, 27)$ and $(31, 4)$, respectively. In both cases $\phi_1 \phi_2 = h^2$, and h^2 is the minimal power of h satisfying the Gluing and Torelli conditions stated in Proposition 3.5.7.

Suppose that $r(S) \in \mathcal{R}_1$. Then $\mathrm{Aut}(S) = \langle f \rangle \cong \mathbb{Z}_2$. By Proposition 5.1.5 and Remark 5.1.6, f^* is the reflection along the line generated by the unique ample class A such that $A^2 = 2$. Taking $W = C$ in the proof of Proposition 5.2.3, our argument there shows that the divisor $A = 4H - C$ is ample and $A^2 = 2$. So $f^*\colon N^1(S)_{\mathbb{R}} \to N^1(S)_{\mathbb{R}}$ is the reflection given by:

$$\alpha \mapsto (A \cdot \alpha)A - \alpha.$$

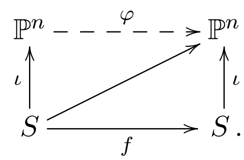
One checks directly that, for each value of $r(S) \in \mathcal{R}_1$, the isometry ϕ represented by the matrix in table (M) coincides with this reflection. □

Remark 5.2.5. In all cases of Proposition 5.2.4, the divisor $D = 4H - C$ is ample. This appeared in the proof for $r(S) \in \mathcal{R}_1$. For $r(S) \in \mathcal{R}_2$ or $\mathcal{R}_3$, D is nef and big, and S contains no rational curves, so

D is automatically ample.

Now that we have explicitly described the action of the generators of $\mathrm{Aut}(S)$ on $\mathrm{Pic}(S)$ when $r(S) \in \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$, we proceed to construct Cremona transformations realizing them. We will use the following criterion to determine when $S \subset \mathbb{P}^3$ is projectively equivalent to its image under a Cremona transformation of $\mathbb{P}^3$.

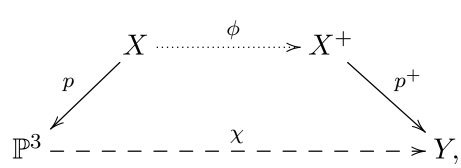
Lemma 5.2.6. Let $\iota\colon S \hookrightarrow \mathbb{P}^n$ be a subvariety embedded by a complete linear system $|H|$. Let $\varphi \in \mathrm{Bir}(\mathbb{P}^n)$ be a Cremona transformation whose restriction to S is an isomorphism onto its image $S' = \varphi(S) \subset \mathbb{P}^n$, and assume that S' is embedded by a complete linear system in $\mathbb{P}^n$. Then S and S' are projectively equivalent in $\mathbb{P}^n$ if and only if there is an automorphism $f \in \mathrm{Aut}(S)$ fitting into a commutative diagram:



Proof. Denote by $H' \in \mathrm{Pic}(S)$ the pullback of the hyperplane class of $\mathbb{P}^n$ under the embedding $\varphi|_S \circ \iota$. By assumption, $\varphi|_S \circ \iota\colon S \hookrightarrow \mathbb{P}^n$ is given by the complete linear system $|H'|$. Hence, the condition that $f \in \mathrm{Aut}(S)$ fits into the commutative diagram above is equivalent to the condition $H' = f^*H$. □

In order to construct Cremona transformations realizing the generators of $\mathrm{Aut}(S)$ described in Proposition 5.2.4, we will consider the Sarkisov links initiated by blowing up the curves $C \subset S$ listed in Proposition 5.2.4. In the case $r(S) = 20$, we will also need a curve $C' \in |4H - C|$, which has genus and degree $(g, d) = (3, 6)$. We recall some numerics of these Sarkisov links, which can be recovered from [CM13, Table 1] and [BL12, Example 4.7(ii)].

Remark 5.2.7 ([CM13, Table 1], [BL12, Example 4.7(ii)])**.** Let $C \subset \mathbb{P}^3$ be a smooth curve of genus g and degree d, where (g, d) is one of the pairs in (C) above or $(g, d) = (3, 6)$. Suppose that C is general in the Hilbert scheme $\mathcal{H}_{g,d}$, so that it satisfies conditions (1) and (2) of Theorem 4.3.5. By Theorem 4.3.5, the blowup $p\colon X \to \mathbb{P}^3$ of C initiates a Sarkisov link $\chi\colon \mathbb{P}^3 \dashrightarrow Y$ fitting into a diagram:



where ϕ is a flop or an isomorphism, Y is a smooth Fano 3-fold with $\rho(Y) = 1$, and $p^+\colon X^+ \to Y$ is the blowup of Y along a smooth curve C^+ of genus g^+ and degree d^+. Here, the degree is measured with respect to the ample generator of $\mathrm{Pic}(Y)$.

Denote by $H \in \mathrm{Pic}(X)$ the pullback of the hyperplane class of $\mathbb{P}^3$, by $H^+ \in \mathrm{Pic}(X^+)$ the pullback of the ample generator of $\mathrm{Pic}(Y)$, and by E and E^+ the exceptional divisors of p and p^+, respectively. With respect to the bases $\{H^+, E^+\}$ of $\mathrm{Pic}(X^+)$ and $\{H, E\}$ of $\mathrm{Pic}(X)$, the isomorphism ϕ^* takes the form

$$\phi^* = \begin{pmatrix} a & \frac{ac-1}{b} \\ -b & -c \end{pmatrix}$$

for suitable integers a, b and c.

For each pair (g, d) in (C) above or $(g, d) = (3, 6)$, the Fano 3-fold Y, as well as the values of g^+, d^+, a, b and c, are displayed in the following table:

(g,d)	$(14,11)$	$(6,9)$	$(10,10)$	$(2,8)$	$(11,10)$	$(3,6)$	$(5,8)$	$(4,8)$	$(3,8)$
Y	$\mathbb{P}^3$	$\mathbb{P}^3$	$\mathbb{P}^3$	$\mathbb{P}^3$	$\mathbb{P}^3$	$\mathbb{P}^3$	$\mathbb{P}^3$	X_5	$\mathbb{P}^3$
(g^+,d^+)	$(14,11)$	$(6,9)$	$(10,10)$	$(2,8)$	$(11,10)$	$(3,6)$	$(5,8)$	$(4,10)$	$(3,8)$
(a,b,c)	$(19,5,19)$	$(27,7,27)$	$(23,6,23)$	$(31,8,31)$	$(11,3,11)$	$(3,1,3)$	$(7,2,7)$	$(11,3,5)$	$(15,4,15)$

(S)

We are now prepared to identify the generators of the automorphism group of an Aut-general smooth quartic surface S with Picard rank 2 and discriminant $r(S) \in \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$ as restrictions of Cremona transformations of $\mathbb{P}^3$. These Cremona transformations will be constructed as compositions of Sarkisov links initiated by blowing up smooth curves $C \subset S$ with invariants (g, d) listed in Table (C). However, for the blowup of C to initiate a Sarkisov link, it is necessary that C satisfies the generality conditions (1) and (2) of Theorem 4.3.5. These conditions are guaranteed when C meets the assumptions of Corollary 4.3.10, specifically $\mathrm{Pic}(S)$ is generated by the classes of hyperplane section H and C, and $4H - C$ is ample on S.

Lemma 5.2.8. Let S be a smooth quartic surface with Picard rank 2 and discriminant $r(S) \in \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$. Let (g, d) be the pair of invariants in Table (C) corresponding to $r(S)$, or $(g, d) = (3, 6)$ if $r(S) = 20$. Then there exists a smooth curve $C \subset S$ of genus and degree (g, d) such that $\mathrm{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}C$, and $4H - C$ is ample on S.

Proof. If (g, d) is one of the pairs of Table (C), then the existence of a smooth curve C of genus and degree (g, d) is guaranteed by Proposition 5.2.4. If $r(S) = 20$ and $(g, d) = (3, 6)$, then $\mathrm{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}W$, where W is a curve of genus and degree $(11, 10)$, again by Proposition 5.2.4. By Proposition 3.5.2, a general element $C \in |4H - W|$ is a smooth curve of genus and degree $(3, 6)$.

We now show that the divisor $D = 4H - C$ is ample. For (g, d) in Table (C) this is Remark 5.2.5; for $(g, d) = (3, 6)$, D is a nef and big divisor and S contains no rational curves (see the proof of Proposition

5.2.3), therefore D is ample. □

Proposition 5.2.9 ($\mathbb{Z}_2$-case)**.** Let S be an Aut-general smooth quartic surface with $\rho(S) = 2$ and discriminant $r(S) \in \{17, 41\}$. Then the restriction homomorphism $\mathrm{Bir}(\mathbb{P}^3; S) \to \mathrm{Aut}(S) \cong \mathbb{Z}_2$ is surjective.

Proof. We first treat the case $r(S) = 41$. By Proposition 5.2.4, $\mathrm{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}C$, where C is a smooth curve of genus and degree $(6, 9)$. Denote by $p\colon X \to \mathbb{P}^3$ the blowup of $\mathbb{P}^3$ along C, and by $\tilde{S} \subset X$ the strict transform of S. By Lemma 5.2.8 and Remark 5.2.7, $X \to \mathbb{P}^3$ initiates a Sarkisov link $\chi\colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$. More precisely, χ fits into a commutative diagram:

$$\begin{array}{ccccc} & X & \overset{\phi_1}{\dashrightarrow} & X^+ & \\ {}^{p}\swarrow & & & & \searrow^{p^+} \\ \mathbb{P}^3 & & \overset{\chi}{\dashrightarrow} & & \mathbb{P}^3. \end{array}$$

Using the notation of Remark 5.2.7, we first prove that χ restricts to an isomorphism on S. Indeed, the restriction of p to $\tilde{S}$ is clearly an isomorphism onto S. Moreover, for any curve $\gamma \subset X$ flopped by ϕ, we have $\tilde{S} \cdot \gamma = -K_X \cdot \gamma = 0$. Since $-K_X|_{\tilde{S}} = 4H - C$ is ample on $\tilde{S}$ by Remark 5.2.5, γ must be disjoint from $\tilde{S}$. Since ϕ preserves anti-canonical sections, the class of $S^+ = \phi(\tilde{S})$ on X^+ is $4H^+ - E^+$. So, for any curve $e^+ \subset X^+$ contracted by p^+, $S^+ \cdot e^+ = 1$, and so p^+ restricts to an isomorphism on S^+. We thus conclude that $\chi|_S\colon S \dashrightarrow \chi(S)$ is an isomorphism.

By Remark 5.2.7, in terms of the bases $\{H^+, E^+\}$ for $N^1(X^+)$ and $\{H, E\}$ for $N^1(X)$, the isomorphism $\phi^*\colon N^1(X^+) \to N^1(X)$ is given by the matrix

$$\phi^* = \begin{pmatrix} 27 & 104 \\ -7 & -27 \end{pmatrix}.$$

Notice that this is the same matrix as the one corresponding to the generator τ of $\mathrm{Aut}(S)$ in Table (M). In particular, $\tau^* H^+ = H$ and thus, by Lemma 5.2.6, $\chi(S)$ is projectively equivalent to S. Up to composing it with an automorphism of $\mathbb{P}^3$, we may assume that $\chi \in \mathrm{Bir}(\mathbb{P}^3; S)$ and $\chi|_S = \tau$. This proves that the restriction homomorphism $\mathrm{Bir}(\mathbb{P}^3; S) \to \mathrm{Aut}(S) = \langle \tau \rangle$ is surjective.

For $r(S) = 17$, we pick $C \subset S$ a smooth curve of genus and degree $(14, 11)$, and follow the exact same argument. The numerics for the corresponding link are given again in Remark 5.2.7. □

Proposition 5.2.10 ($\mathbb{Z}_2 * \mathbb{Z}_2$-case)**.** Let S be an Aut-general smooth quartic surface with $\rho(S) = 2$ and discriminant $r(S) \in \{28, 56\}$. Then the restriction homomorphism $\mathrm{Bir}(\mathbb{P}^3; S) \to \mathrm{Aut}(S) \cong \mathbb{Z}_2 * \mathbb{Z}_2$ is surjective.

Proof. We first treat the case $r(S) = 56$. By Proposition 5.2.4, $\mathrm{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}C_1$, where C_1 is a smooth curve of genus and degree $(2, 8)$. Moreover, $\mathrm{Aut}(S) \cong \mathbb{Z}_2 * \mathbb{Z}_2$ is generated by two involutions

τ_1 and τ_2 that act on $\mathrm{Pic}(S)$ as

$$\tau_1^* = \begin{pmatrix} 31 & 120 \\ -8 & -31 \end{pmatrix} \quad \text{and} \quad \tau_2^* = \begin{pmatrix} -1 & 0 \\ 8 & 1 \end{pmatrix}$$

with respect to the basis $\{H, C_1\}$.

Denote by X_1 the blowup of $\mathbb{P}^3$ along C_1. By Lemma 5.2.8 and Remark 5.2.7, $X_1 \to \mathbb{P}^3$ initiates a Sarkisov link $\chi_1 \colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$. Arguing exactly as in the proof of Proposition 5.2.9, we see that χ_1 restricts to an isomorphism on S and, after composing it with an automorphism of $\mathbb{P}^3$, we may assume that $\chi_1(S) = S$ and $\chi_1|_S = \tau_1$.

As for the second generator τ_2 of $\mathrm{Aut}(S)$, let C_2 be a smooth element of $|4H - C_1|$. Then C_2 is also of genus and degree $(2, 8)$, and so the blowup $X_2 \to \mathbb{P}^3$ along C_2 again initiates a Sarkisov link $\chi_2 \colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$. As before, after composing it with an automorphism of $\mathbb{P}^3$, we may assume that $\chi_2(S) = S$, and the induced automorphism on S acts on $\mathrm{Pic}(S)$ as

$$\begin{pmatrix} 31 & 120 \\ -8 & -31 \end{pmatrix}$$

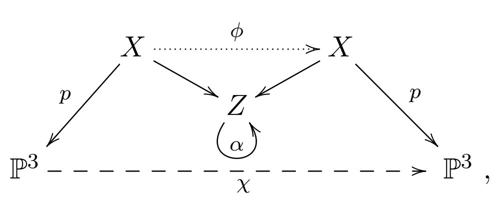
with respect to the basis $\{H, C_2\}$. Changing the basis to $\{H, C_1\}$, we get

$$\chi_2|_S^* = \begin{pmatrix} 1 & 4 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 31 & 120 \\ -8 & -31 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 0 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 0 \\ 8 & 1 \end{pmatrix}.$$

So $\chi_i|_S = \tau_i$ for $i = 1, 2$, and we get a surjection $\mathrm{Bir}(\mathbb{P}^3; S) \to \mathrm{Aut}(S) = \langle \tau_1, \tau_2 \rangle$.

The case $r(S) = 28$ is analogous: we choose C_1 to be a smooth curve of genus and degree $(10, 10)$ and C_2 a smooth element of $|5H - C_1|$. Then C_2 is also a curve of genus and degree $(10, 10)$, and the construction above works verbatim. □

Remark 5.2.11. The four links described in the proofs of Propositions 5.2.9 and 5.2.10, initiated by the blowup of $\mathbb{P}^3$ along smooth curves of genus and degree $(6, 9)$, $(14, 11)$, $(2, 8)$ and $(10, 10)$ were described in detail in [Zik23a, Proposition 3.1 and Remark 3.2]. Each one is a birational involution $\chi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ fitting into a commutative diagram:



where $\phi \colon X \dashrightarrow X$ is a flop, the anti-canonical model Z of X is a double cover of $\mathbb{P}^3$ ramified along a sextic hypersurface, and $\alpha \colon Z \to Z$ is the the deck transformation of Z over $\mathbb{P}^3$.

Proposition 5.2.12 ($\mathbb{Z}$-case)**.** Let S be an Aut-general smooth quartic surface with $\rho(S) = 2$ and discriminant $r(S) \in \{20, 32, 40, 48\}$. Then the restriction $\mathrm{Bir}(\mathbb{P}^3; S) \to \mathrm{Aut}(S) \cong \mathbb{Z}$ is surjective.

Proof. We first treat the case $r(S) = 40$. By Proposition 5.2.4, there is a smooth curve $C_1 \subset S$ of genus and degree $(4, 8)$ such that $\mathrm{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}C_1$. By Lemma 5.2.8 and Remark 5.2.7, the blowup $p_1 \colon X_1 \to \mathbb{P}^3$ along C_1 initiates a Sarkisov link $\chi_1 \colon \mathbb{P}^3 \dashrightarrow X_5$ that fits into a commutative diagram:

$$\begin{array}{ccccc} & X_1 & \overset{\phi_1}{\dashrightarrow} & X_1^+ & \\ {}^{p_1}\swarrow & & & & \searrow^{p_1^+} \\ \mathbb{P}^3 & & \overset{\chi_1}{\dashrightarrow} & & X_5, \end{array}$$

where the smooth curve $C_1^+ \subset X_5$ is the center of the blowup $p_1^+ \colon X_1^+ \to X_5$, with degree and genus $(4, 10)$. Recall that the degree is measured with respect to the ample generator of $\mathrm{Pic}(X_5)$. Set $S_1 := \chi_1(S) \subset X_5$. Arguing as in the proof of Proposition 5.2.9, we see that $\chi_1|_S \colon S \to S_1$ is an isomorphism. Set $\sigma_1 := \chi_1|_S \colon S \to S_1$. By Remark 5.2.7, with respect to the bases $\{H^+, C_1^+\}$ of $\mathrm{Pic}(S_1)$ and $\{H, C_1\}$ of $\mathrm{Pic}(S)$, σ_1^* takes the form

$$\sigma_1^* = \begin{pmatrix} 11 & 18 \\ -3 & -5 \end{pmatrix}.$$

Using Lemma 5.2.6, one can check that S and S_1 are not projectively equivalent.

We will now perform a second link. First, note that the surface S_1 is a smooth anticanonical surface in X_5, i.e., $S_1 \in |-K_{X_5}|$. After identifying S_1 with its strict transform in X_1^+, it follows that $-K_{X_1^+}|_{S_1} = 2H^+ - C_1^+$ and it is an ample class on S_1. Hence, similar to Proposition 4.3.8, Proposition 4.3.9 and Corollary 4.3.10, the blowup of X_5 along any smooth curve in $|2H^+ - C_1^+|$ is weak Fano and it initiates a Sarkisov link from X_5. Indeed, any general smooth curve $C_2 \in |2H^+ - C_1^+|$ has degree and genus $(4, 10)$, and so, the Sarkisov link initiated by the blowup of X_5 along C_2 is the inverse of a Sarkisov link initiated by the blowup of $\mathbb{P}^3$ along a smooth curve C_2^+ of genus and degree $(4, 8)$ in Remark 5.2.7.

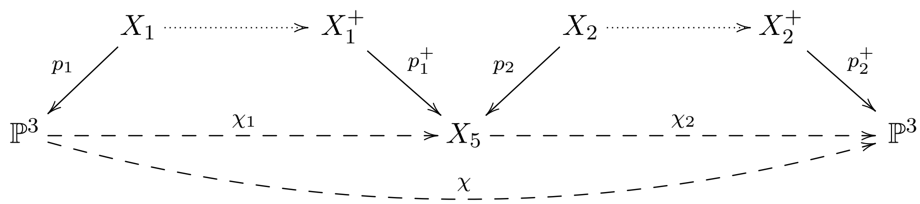
Thus, let $C_2 \subset S_1$ be a smooth element of the linear system $|2H^+ - C_1^+|$. Denote by $p_2 \colon X_2 \to X_5$ the blowup of X_5 along C_2, and $\chi_2 \colon X_5 \dashrightarrow \mathbb{P}^3$ the Sarkisov link initiated from it. Once again, we follow the notation introduced in Remark 5.2.7. We denote by $E_2^+ \subset X_2^+$ the exceptional divisor of the blowup $p_2^+ \colon X_2^+ \to \mathbb{P}^3$, by $C_2^+ := p_2^+(E_2^+) \subset \mathbb{P}^3$ its center, and set $S_2 := \chi_2(S_1) \subset \mathbb{P}^3$. Arguing as in the proof of Proposition 5.2.9, we see that the restriction $\chi_2|_{S_1} \colon S_1 \to S_2$ is an isomorphism. Set $\sigma_2 := \chi_2|_{S_1} \colon S_1 \to S_2$, and write A and A^+ for the hyperplane classes of S_1 and S_2, respectively. By Remark 5.2.7, with respect to the bases $\{A^+, C_2^+\}$ of $\mathrm{Pic}(S_2)$ and $\{A, C_2\}$ of $\mathrm{Pic}(S_1)$, σ_2^* takes the form

$$\sigma_2^* = \begin{pmatrix} 5 & 18 \\ -3 & -11 \end{pmatrix}.$$

A smooth element $C' \subset S_2$ in the linear system $|4A^+ - C_2^+|$ is a curve of genus and degree $(4, 8)$. Computing the matrix of the composition $(\sigma_2 \circ \sigma_1)^*\colon \operatorname{Pic}(S_2) \to \operatorname{Pic}(S)$ with respect to the bases $\{A^+, C'\}$ of $\operatorname{Pic}(S_2)$ and $\{H, C_1\}$ of $\operatorname{Pic}(S)$, we get:

$$(\sigma_2 \circ \sigma_1)^* = \begin{pmatrix} 11 & 18 \\ -3 & -5 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 5 & 18 \\ -3 & -11 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 0 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 43 & 18 \\ -12 & -5 \end{pmatrix}.$$

Notice that this is the same matrix as the one corresponding to the generator of $\operatorname{Aut}(S)$ in Table (M). By Lemma 5.2.6, after composing it with an automorphism of $\mathbb{P}^3$, we may assume that $(\chi_2 \circ \chi_1)(S) = S$. Thus, the composition $\chi = \chi_2 \circ \chi_1$ restricted to S generates $\operatorname{Aut}(S)$.



The cases $r(S) = 20, 32, 48$ are analogous: the birational map $\chi\colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ that stabilizes S and generates $\operatorname{Aut}(S) \cong \mathbb{Z}$ is always the composition of two Sarkisov links χ_1 and χ_2. In these cases, both Sarkisov links are Cremona transformation of $\mathbb{P}^3$. The first Sarkisov link $\chi_1\colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ is initiated by the blowup of $\mathbb{P}^3$ along a smooth curve $C_1 \subset S$ of genus and degree (g, d) indicated in the table below, while the second Sarkisov link $\chi_2\colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ is initiated by the blowup of $\mathbb{P}^3$ along a smooth curve $C_2 \subset \chi_1(S)$ such that $C_2 \sim 4H^+ - C_1^+$. The curve C_2 has genus and degree (g', d') listed in the table below.

$r(S)$	(g, d)	(g', d')
20	$(11, 10)$	$(3, 6)$
32	$(5, 8)$	$(5, 8)$
48	$(3, 8)$	$(3, 8)$

(Z)

□

We are now ready to prove Theorem B:

Theorem 5.2.13 (Theorem B)**.** Let $S \subset \mathbb{P}^3$ be a smooth quartic surface with Picard rank $\rho(S) = 2$.

(1) If $r(S) > 57$ or $r(S) = 52$, then $\operatorname{Bir}(\mathbb{P}^3; S) = \{1\}$.

(2) If $r(S) \leq 57$, $r(S) \neq 52$ and S is Aut-general, then the restriction map $\Psi\colon \operatorname{Bir}(\mathbb{P}^3; S) \to \operatorname{Aut}(S)$ is surjective.

Proof. First suppose that $r(S) > 57$ or $r(S) = 52$. Then $\mathrm{Bir}(\mathbb{P}^3; S) = \mathrm{Aut}(\mathbb{P}^3; S) = \{1\}$ by Corollary 5.2.2.

If $r(S) \leq 57$, $r(S) \neq 52$ and $\mathrm{Aut}(S) \neq \{1\}$, then $r(S) \in \{17, 41\} \cup \{28, 56\} \cup \{20, 32, 40, 48\}$ by Proposition 5.2.3. For $r(S)$ in each one of these three sets, we conclude by Propositions 5.2.9, 5.2.10 and 5.2.12, respectively. □

Remark 5.2.14. The generality hypothesis in part two of Theorem 5.2.13 is necessary in order to determine the entire automorphism group $\mathrm{Aut}(S)$ and to find its generators by using Proposition 3.5.7. However, when $r(S) < 57$, $r(S) \neq 52$ and $\mathrm{Aut}(S)$ if finite, i.e., $\mathrm{Aut}(S) = \{1\}$ or $\mathrm{Aut}(S) = \mathbb{Z}_2$, this assumption can be removed. Indeed, when the finite index subgroup $\mathrm{Aut}^{\pm}(S) \subset \mathrm{Aut}(S)$ of a smooth quartic $S \subset \mathbb{P}^3$ is finite, we have that $\mathrm{Aut}(S)$ is finite. Hence, any non-trivial automorphism of S is an involution and so, $\mathrm{Aut}^{\pm}(S) = \mathrm{Aut}(S)$.

5.3 Insights for higher Picard rank: An example

A natural extension of our work is to investigate Gizatullin's problem for quartics with higher Picard rank. In this section, we analyze Oguiso's example (Example 5.0.3), which is the only known example for Picard rank ≥ 3 where Problem 1 is positively solved. We try to realize the automorphisms of the quartic surface as Cremona transformations constructed from Sarkisov links. The conclusions of this section emerged from discussions with Carolina Araujo, Michela Artebani, Cesar Huerta and Manuel Leal at the V Latin American School of Algebraic Geometry (V ELGA).

Let us first recall some facts on elliptic curves. Let $E \subset \mathbb{P}^2$ be an elliptic curve. It is known that E has the structure of a group, on which there is a specified point O_E playing the role of the identity element of the group. From this group structure we get some natural automorphisms of E: the *inversion* and the *translations*. The inversion is the map $\iota_E \colon E \to E$ given by $p \mapsto -p$, and a translation by a fixed point $q \in E$ is the map $t_q \colon E \to E$ given by $p \mapsto p + q$. Note that the automorphisms ι_E and $\tau_q := \iota_E \circ t_q$ are involutions of E. A proof of the following Lemma is done by exploiting the Weierstrass form of E.

Lemma 5.3.1 ([Ogu12, Theorem 2.2])**.** Let $E \subset \mathbb{P}^2_{\mathbf{k}}$ be an elliptic curve over an arbitrary field $\mathbf{k}$. Then any automorphism of E is the restriction of a Cremona transformation of $\mathbb{P}^2_{\mathbf{k}}$.

Now, we describe Example 5.0.3 in more details, following [Ogu12].

Let $S \subset \mathbb{P}^3$ be a smooth quartic surface with Picard lattice $\mathrm{Pic}(S)$ of rank 3, generated by a hyperplane section H and two skew lines $L, M \subset \S$. The matrix associated to the intersection product on S, with respect to this basis, is the matrix

$$\begin{pmatrix} 4 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{pmatrix}. \tag{5.3}$$

By looking for elements $D \in \mathrm{Pic}(S)$ such that $H \cdot D = 1$ and $D^2 = -2$, one can conclude that the only two lines on S are L and M. Moreover, each line determines an elliptic fibration on S. Indeed, consider the pencil of planes $P \subset \mathbb{P}^3$ containing L. For each such plane P, $P \cap S = L \cup E_P$, where E_P is a plane cubic curve. Note that $E_P \in |H - L|$, and so $E_P^2 = (H-L)^2 = 0$ and $E_P \cdot M = (H-L) \cdot M = 1$ on S. Hence, the linear system $|H - L|$ defines an elliptic fibration $\phi_{|H-L|} \colon S \to \mathbb{P}^1$ with section M. Similarly, we have the elliptic fibration $\phi_{|H-M|} \colon S \to \mathbb{P}^1$ with section L. By definition of an elliptic fibration, the generic fiber of it is an elliptic curve, i.e., a smooth curve of genus one. We can verify that all the fibers of both fibrations are irreducible.

For the elliptic fibration $\phi_{|H-L|}$, we regard M as the zero section, i.e., for each elliptic fiber E_P, the point $M \cap E_P \in E_P$ is the distinguish point O_{E_P}. For any section $C \subset S$ of $\phi_{|H-L|}$, it follows that $C \cdot E_P = 1$ for each fiber E_P. Thus, we denote by $C_{E_P} \in E_P$ the intersection point of the section C with the fiber E_P. Therefore, we can define an involution ι_L of S whose restriction to each elliptic fiber E_P is the inversion ι_{E_P}, and an automorphism t_C of S whose restriction to each elliptic fiber E_P is the translation $t_{C_{E_P}}$. It can be shown that the set of sections of the elliptic fibration is the set

$$\{C_n \in \mathrm{Pic}(S) | C_n = (10n^2 - 6n)H - (10n^2 - 7n)L - (3n-1)M,\ n \in \mathbb{Z}\}.$$

Furthermore, the elliptic fibration $\phi_{|H-L|}$ is the restriction to S of the linear projection $\mathbb{P}^3 \to \mathbb{P}^1$ from L. The fibers of this linear projection are precisely the planes $P \cong \mathbb{P}^2$ containing the line L, and the restriction of these fibers to S are the fibers E_P of $\phi_{|H-L|}$. The restrictions of ι_L and t_C to each generic fiber E_P are ι_{E_P} and $t_{C_{E_P}}$, respectively, and these are induced by Cremona transformations of the fiber $P \cong \mathbb{P}^2$ by Lemma 5.3.1. It follows that ι_L and t_C are induced by Cremona transformations of $\mathbb{P}^3$. Similarly, once we regard L as the zero section of the elliptic fibration $\phi_{|H-M|}$, we obtained the map ι_M corresponding to the inversion in each elliptic fiber of $\phi_{|H-M|}$, which is again, induced by a Cremona transformation of the ambient space $\mathbb{P}^3$.

By describing a fundamental domain for the action of the free product $\langle \iota_L, \iota_M, \tau \rangle \subset \mathrm{Aut}(S)$, where $\tau := \iota_L \circ t_{C_1}$ and t_{C_1} is the translation by the section C_1 of the elliptic fibration $\phi_{|H-L|}$, Oguiso proved that these three involutions generate the automorphism group of $S \subset \mathbb{P}^3$, i.e., $\mathrm{Aut}(S) = \langle \iota_L, \iota_M, \tau \rangle \cong \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$. Therefore, every automorphism of S is the restriction of a birational map of $\mathbb{P}^3$. However, for a more precisely description of such Cremona maps of $\mathbb{P}^3$ it is necessary to first find the Weierstrass form of each elliptic fibration, and then describe the birational maps of $\mathbb{P}^2$ realizing the inversions and translations.

Our purpose is to describe the Cremona transformations of $\mathbb{P}^3$ that realize the generators ι_L, ι_M and τ of the automorphism group of S from the Sarkisov links point of view as we did in previous Section 5.2. Hence, we start by describing the action of these generators on the Picard lattice $\mathrm{Pic}(S)$, with respect to the basis $\{H, L, M\}$:

$$\iota_L^* = \begin{pmatrix} 13 & 12 & 0 \\ -14 & -13 & 0 \\ 6 & 6 & 1 \end{pmatrix}, \quad \iota_M^* = \begin{pmatrix} 13 & 0 & 12 \\ 6 & 1 & 6 \\ -14 & 0 & -13 \end{pmatrix}, \quad \text{and} \quad \tau^* = \begin{pmatrix} 5 & 4 & 4 \\ -3 & -2 & -3 \\ -3 & -3 & -2 \end{pmatrix}. \tag{5.4}$$

Each of the involutions above have an invariant lattice with rank two. Indeed, each invariant lattice is given by

$$H^2(S,\mathbb{Z})^{\iota_L} = \langle H - L, M\rangle, \quad H^2(S,\mathbb{Z})^{\iota_M} = \langle H - M, L\rangle \quad \text{and} \quad H^2(S,\mathbb{Z})^{\tau} = \langle H - L, H - M\rangle.$$

Observe that the two classes of curves $C_1 = 2H + L + M$ and $C_2 = 3H - L - M$ on $S \subset \mathbb{P}^3$ are such that $H \cdot C_i = 10$ and $C_i^2 = 20$. Thus, both of them correspond to classes of curves of type $(11, 10)$. Note that $\mathrm{Pic}(S) = \langle H, C_1, H + L\rangle = \langle H, C_2, H + L\rangle$. Define now the class $C_i' = 4H - C_i$. It is follows that $C_1' = 2H - L - M$ and $C_2' = H + L + M$ are curves of type $(3, 6)$. If we consider the primitive sublattice $L := \langle H, C_1\rangle \subset \mathrm{Pic}(S)$, we conclude that L is isomorphic to the Picard lattice of any smooth quartic surface $S' \subset \mathbb{P}^3$ with $\rho(S') = 2$ and $r(S') = 20$, from Proposition 5.2.4. For an Aut-general such surface S', the automorphism group $\mathrm{Aut}(S') \cong \mathbb{Z}$ is generated by an infinite order automorphism and it is the restriction of a Cremona transformation φ of $\mathbb{P}^3$ which is obtained as follows. The birational map φ is the composition of a Sarkisov link initiated by the blowup of a curve of type $(11, 10)$ with a Sarkisov link initiated by the blowup of a curve of type $(3, 6)$ (see Proposition 5.2.12 and table (Z)).

Proposition 5.3.2. Let $S \subset \mathbb{P}^3$ a smooth quartic surface with Picard rank $r(S) = 3$ and intersection matrix given by the matrix (5.3).

1. The only rational curves on S with degree ≤ 10 are L and M.
2. A general member of the linear systems $|2H + L + M|$, $|3H - L - M|$ and $|2H - L - M|$ is smooth.
3. Let $C_1 \in |2H + L + M|$ be general. Then $K_{C_1} = \mathcal{O}_{C_1}(2)$.
4. Let $C_2 \in |3H - L - M|$ be general. Then C_2 is contained in a unique cubic surface T.

Proof. Let $\Gamma \subset \mathrm{Pic}(S)$ be a rational curve. Write $\Gamma = aH + bL + cM$ for some integers a, b, c. Then, $-2 = \Gamma^2 = \Gamma^2 = 4a^2 - 2b^2 - 2c^2 + 2ab + 2ac = -2$. Define $\delta := \deg(\Gamma) = H \cdot \Gamma = 4a + b + c > 0$. By substituting c by $\delta - 4a - b$ in Γ^2 we get

$$-2 = \Gamma^2 = -36a^2 - 16ab - 4b^2 + 18a\delta + 4b\delta - 2\delta^2. \tag{5.5}$$

By varying δ between 1 and 10 and looking for the integer solutions of equation (5.5), we get that the only possibilities are $\Gamma = L$ or $\Gamma = M$. This gives (1).

Now, to prove (2), we will see that the linear systems have no fixed components and so the assertion follows from 3.1.8(1). Consider the linear system $|2H - L - M|$ and assume by contradiction that it has a rational curve $\Gamma = aH + bL + cM$ as fixed component, for $a, b, c \in \mathbb{Z}$. The curve Γ is such that and $0 < \deg(\Gamma) = H \cdot \Gamma < H \cdot (2H - L - M) = 6$ and so the only possibilities are $\Gamma = L$ or $\Gamma = M$ by part (1). This implies that $2H - L - M$ is nef since $(2H - L - M) \cdot L = 4 = (2H - L - M) \cdot M$. By Proposition 3.1.8(5), $2H - L - M = \alpha E + \Gamma$, where E is an irreducible curve with $p_a(E) = 1$ and $\alpha \geq 2$; and by [SD74, (2.7.3),(2.7.4)], $E \cdot \Gamma \in \{0, 1\}$. Thus, $4 = (2H - L - M) \cdot \Gamma = (\alpha E + \Gamma) \cdot \Gamma \leq \alpha - 2$. It follows that $\alpha = 6$ and so $4 = (2H - L - M)^2 = (6E + \Gamma)^2 \leq 12 - 2 = 10$. This is not possible. Therefore, $|2H - L - M|$ has no fixed component. Similarly, we conclude that the linear systems $|2H + L + M|$ and $|3H - L - M|$ has no fixed part.

Note that $K_{C_1} = C_1|_{C_1} = (2H + L + M)|_{C_1} = 2H|_{C_1}$ by adjunction formula and the fact that $C_1 \cdot L = 0 = C_1 \cdot M$. This is (3). To see that (4) holds we observe the following. From the structural sequence of $C_1 \subset \mathbb{P}^3$ we get that $h^0(\mathbb{P}^3, \mathcal{I}_{C_1}(3)) \geq h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)) - h^0(C_1, \mathcal{O}_{C_1}(3H)) = 10 - h^0(C_1, \mathcal{O}_{C_1}(3H))$. Using the Riemann-Roch theorem on C_1, we conclude that $h^0(\mathbb{P}^3, \mathcal{I}_{C_1}(3)) \geq 1$, i.e., C_1 is contained in a cubic surface T, which is unique by degree reasons. □

From now on we consider the curves $C_1 \in |2H + L + M|$ and $C_1' \in |3H - L - M|$ of type $(11, 10)$, and the curve $C_2 \in |2H - L - M|$ of type $(3, 6)$ to be smooth. The curves C_1 and C_2 belong to the Hilbert scheme $\mathcal{H}_{11,10}$ that parametrized smooth curves $C \subset \mathbb{P}^3$ of genus 11 and degree 10. These curves are not general in $\mathcal{H}_{11,10}$. Indeed, the set of curves $C \in \mathcal{H}_{11,10}$ that are contained in a cubic surface defines a closed loci in $\mathcal{H}_{11,10}$. The same is true for curves $C \in \mathcal{H}_{11,10}$ such that $K_C = \mathcal{O}_C(2)$.

Proposition 5.3.3. Let $C, C' \in \mathcal{H}_{11,10}$ be smooth curves in $\mathbb{P}^3$ of type $(11, 10)$. Assume that C is general and that $K_{C'} = \mathcal{O}_{C'}(2)$.

1. A very general quartic surface containing C has Picard rank $r(S) = 2$, discriminant $r(S) = 20$ and the intersection matrix is given by
$$\begin{pmatrix} 4 & 10 \\ 10 & 20 \end{pmatrix}.$$

2. A very general quartic surface containing C' has Picard rank $r(S) = 3$ and the intersection matrix is given by
$$\begin{pmatrix} 4 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{pmatrix}.$$

Proof. We start by proving (1) Let $S \subset \mathbb{P}^3$ be a smooth quartic surface containing C. Consider the sublattice $L = \langle H, C\rangle$ of $\mathrm{Pic}(S)$, where H denotes a hyperplane section. It is possible to show that there exist no elements $x \neq 0$ in the discriminant group $A(L) \cong \mathbb{Z}_2 \times \mathbb{Z}_{10}$ such that $q_L(x) = 0$, where q_L is the quadratic form induced by the bilinear form on L. From [Nik80, Proposition 1.4.1], L is a primitive lattice of $\mathrm{Pic}(S)$. Thus, L is a primitive lattice of the K3 lattice Λ_{K3} with signature $(1, 1)$. Therefore, a very general element in the moduli space of L-polarized K3 surfaces correspond to a smooth quartic surface with $\mathrm{Pic}(S) \cong L$.

Now, consider $S \subset \mathbb{P}^3$ a smooth quartic surface containing C' and denote by H a hyperplane section. Note that $h^0(S, 2H) = 10$, $h^0(S, 2H - C') = 0$ and $h^0(C', 2H - C') = 1$ since $2H$ is very ample, $(2H - C') \cdot H = -2 < 0$, and $K_{C'} = \mathcal{O}_{C'}$. From the structural sequence of $C' \subset S$, we have $11 = h^0(C', 2H) \leq h^0(S, 2H) + h^1(S, 2H - C') = 10 + h^1(S, 2H - C')$. Then $h^1(S, 2H - C') = 1$. Now, from Riemman-Roch we get that $h^0(S, C' - 2H) = 1$. Furthermore, $(C' - 2H)^2 = -4$ which implies that the section of $C' - 2H$ is the union of at least two rational curves, but since $(C' - 2H) \cdot H = 2$, we get that it is precisely the union of two skew lines L, M. Set $L = \langle H, L, M\rangle \subset \mathrm{Pic}(S)$. Again, looking at the discriminant we conclude that L is a primitive sublattice of $\mathrm{Pic}(S)$ and so a very general surface in the moduli of L-polarized K3 surface is a smooth quartic surface with $\mathrm{Pic}(S) \cong L$. □

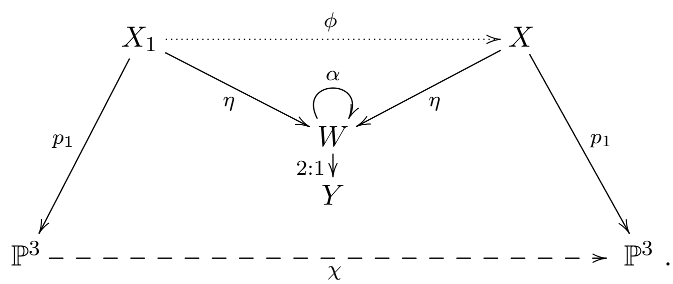
We now investigate the blow-ups of $\mathbb{P}^3$ along the curves C_1 and C_2 on S.

Proposition 5.3.4. Let $S \subset \mathbb{P}^3$ be a smooth quartic surface with $r(S) = 3$ and intersection matrix given by (5.3). Let C_1 be a smooth curve in the linear system $|2H + L + M|$ and denote by $p_1 \colon X_1 \to \mathbb{P}^3$ the blowup X_1 of $\mathbb{P}^3$ along C_1. Then X_1 is weak Fano and $p_1 \colon X_1 \to \mathbb{P}^3$ initiates a Sarkisov link χ, such that $\chi \in \mathrm{Bir}(\mathbb{P}^3; S)$ and $\chi|_S = \tau$. Here τ is the third generator of $\mathrm{Aut}(S)$ in (5.4).

Proof. Consider the class $4H - C_1 = 2H - L - M$. By the proof of Proposition 5.3.2(2), $4H - C_1$ is nef, and so it is big since $(4H - C_1)^2 = 4$. Now, we prove that $4H - C_1$ is ample. If not, there is a rational curve $\Gamma = aH + bL + cM$ that satisfies conditions $\Gamma \cdot (4H - C_1) = 6a + 4b + 4c = 0$ and $\Gamma^2 = 4a^2 - 2b^2 - 2c^2 + 2ab + 2ac$. These two last condition lead to the quadratic equation $-4 = -7a^2 - 8b^2 - 12ab$, which has no integer solutions. Therefore, the variety X_1 is weak Fano and $p_1 \colon X_1 \to \mathbb{P}^3$ initiates a Sarkisov link by Proposition 4.3.9.

Now we describe the Sarkisov links to conclude the second part of the statement. Note that there exists a smooth quadric $Q \subset \mathbb{P}^3$ such that $Q \cap S = C_1' \cup L \cup M$. Consider the class of $C_1' = af_1 + bf_2$ in $\mathrm{Pic}(Q) \cong \mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)$, where f_1 and f_2 are the respective fibers with respect to the two projections $\pi_1, \pi_2 \colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$. Recall that the intersection product on $\mathbb{P}^1 \times \mathbb{P}^1$ is defined by $D \cdot D' = (cf_1 + df_2) \cdot (c'f_1 + d'f_2) = cd' + c'd$, a hyperplane section of $Q \subset \mathbb{P}^3$ has class $f_1 + f_2$ and the canonical divisor $K_{\mathbb{P}^1 \times \mathbb{P}^1} = -2f_1 - 2f_2$. Hence, $6 = \deg(C_1') = a + b$ and $4 = 2g(C_1') - 2 =$

$C_1' \cdot (C_1' + K_{\mathbb{P}^1\times\mathbb{P}^1}) = 2ab - 2a - 2b$. Then, $(a,b) = (2,4)$ or $(a,b) = (4,2)$. Assume without loss of generality that $C_1' = 2f_1 + 4f_2$. The restriction π_2 to C_1' gives a surjective map $C_1' \to \mathbb{P}^1$ of degree 2. Therefore, the curve C_1' is a hyperelliptic. Let $\tilde{S}$ and $\tilde{C}_1'$ be the strict transform of S and C_1' under the blowup $p_1\colon X_1 \longrightarrow \mathbb{P}^3$, respectively. Denote by E the exceptional divisor of p_1, and by H a hyperplane in $\mathbb{P}^3$, its pullback to X_1 and its restriction S. Thus $-K_{X_1} = 4H - E$, $S \cong \tilde{S}$ and $(-K_{X_1})|_{\tilde{C}_1'} = (4H-E)|_{\tilde{C}_1'} = (4H - C_1)|_{C_1'} = C_1'|_{C_1'} = K_{C_1'}$. Furthermore, since the linear system $|4H - E|$ is 5-dimensional, for a generic point $q \in X$, we can consider a pencil of $|4H - E|$ passing through q, generated by elements $D_1, D_2 \in |4H - E|$. The hypersurfaces D_1 and D_2 correspond to smooth quartic surfaces S_1 and S_2 on $\mathbb{P}^3$ containing the curve C_1. We assume these surfaces are very general surfaces containing the curve C_1. Thus, $\mathrm{Pic}(S_1) \cong \mathrm{Pic}(S)$ and $S_1 \cap S_2 = C_1 \cup C_{12}'$, where C_{12}' is a smooth hyperelliptic curve of type $(3,6)$, $K_{C_{12}'} = (-K_{X_1})|_{\tilde{C}'_{12}}$ and $q \in C_{12}'$. It follows that X is covered by such smooth hyperelliptic curves C of type $(3,6)$ such that $(-K_{X_1})|_C = K_C$. Since the map given by $|K_C|$ has degree 2 on every hypereliptic C, we conclude that the anticanonical morphism $\varphi_{|-K_{X_1}|}$ also has degree 2. The anticanonical map $\varphi_{|-K_{X_1}|}\colon X_1 \to Y$ factors through a morphism $X_1 \to W$ to anticanonical model W of X_1 and the map $W \to Y$ is also of degree 2. Thus, we can describe the Sarkisov link initiated by the blowup $p_1\colon X_1 \to \mathbb{P}^3$ by the following diagram



More precisely, $\chi = (\eta \circ p_1^{-1})^{-1} \circ \alpha \circ (\eta \circ p_1^{-1})$. This description guarantees that the smooth quartic surface S is stabilized by χ. It indeed stabilizes any quartic surface containing the curve C_1. Note that ϕ^* is an isometry of order two of $\mathrm{Pic}(X_1) = \langle H, E\rangle$ which stabilizes $-K_{X_1} = 4H - E$. Fix now the basis $H, -K_{X_1}$ of $\mathrm{Pic}(X_1)$. In this basis, the isometry ϕ^* is given by

$$\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}.$$

Since $(\phi^*)^2 = \mathrm{id}$, $a^2 = 1$ and $ab + a = 0$, we have $a = -1$. From 2.2.6, $H \cdot (-K_{X_1})^2 = 6$ and so $5 = \phi^* H \cdot (-K_{X_1})^2 = (-H - bK_{X_1}) \cdot (-K_{X_1})^2 = -6 + b(-K_{X_1})^3 = -6 + 4b$. It follows that $b = 3$ and $\phi^* H = -H + 3(-K_{X_1})$.

On the other hand, the restriction $\sigma = \chi|_S \in \mathrm{Aut}(S)$ is an automorphism of order 2. Note that since $\phi^*(-K_{X_1}) = -K_{X_1}$ and $\phi^* H = -H + 3(-K_{X_1})$, it follows that $\sigma^* C_1' = C_1'$ and $\sigma^* H = -H + 3C_1' = -H + 3(2H - L - M) = 5H - 3L - 3M$. Now, observe that $C_1' = (H - L) + (H - M)$. We claim that both

$H-L$ and $H-M$ are preserved by σ^*. Indeed, assume $\sigma^*(H-L) = aH+bL+cM$ and $\sigma^*(H-M) = a'H+b'L+c'M$. Since $2H-L-M = \phi^*(2HL-M) = \phi^*(aH+bL+cM)+\phi^*(a'H+b'L+c'M)$, we get that $a+a' = 2$, $b+b' = -1$ and $c+c' = -1$. Moreover, from the system of equations

$$\begin{cases} 0 = (H-L)^2 = \sigma^*(H-L) = 4a^2-2b^2-2c^2+2ab+2ac \\ 3\ H\cdot(H-L) = (5H-3L-3M)\cdot(aH+bL+cH) = 14a+11b+11c \\ 2 = a+a' \\ -1 = b+b' \\ -1 = c+c' \end{cases},$$

we get that $a = 1 = a'$, $b' = c = 0$ and $b = c' = -1$. Therefore $\phi^*L = \phi*(H-(H-L)) = 5H-3L-3M-H+L = 4H-2L-3M$ and $\phi^*M = \phi*(H-(H-M)) = 5H-3L-3M-H+M = 4H-3L-2M$. This implies then that the automorphism σ coincides with the automorphism τ in (5.4). □

We could expect that the blowup $p_2\colon X_2 \to \mathbb{P}^3$ of $\mathbb{P}^3$ along the curve C_2 of type $(11,10)$ initiates a Sarkisov link which induces also an automorphism of S. However, this is not the situation.

Proposition 5.3.5. Let $S \subset \mathbb{P}^3$ be a smooth quartic surface with $r(S) = 3$ and intersection matrix given by (5.3). Let C_2 be a smooth curve in the linear system $|3H-L-M|$ and denote by $p_2\colon X_2 \to \mathbb{P}^3$ the blowup X_2 of $\mathbb{P}^3$ along C_2. Then X_2 is not weak Fano and $p_2\colon X_2 \to \mathbb{P}^3$ is not a rank 2 fibration.

Proof. The fact that X_2 is not weak Fano follows from the fact that the class $4H-C_2 = H+L+M$ is not nef: $(4H-C_2)\cdot L = -1 = (4H-C_2)\cdot M$, and by Proposition 4.3.8. To see that $X_2 \to \mathrm{Spec}(\mathbb{C})$ is not a rank 2 fibration, we will find infinitely many curves on X_2 intersecting trivially $-K_{X_2}$ and so the assertion follows from [Zik23b, Proposition 3.15].

Recall that C_2 is contained in a unique smooth cubic surface $T \subset \mathbb{P}^3$. After describing T as the blowup of six general points in $\mathbb{P}^2$, the class of C_2 on the cubic T is $10l-4e_1-4e_2-4e_3-3e_4-3e_5-2e_6$, where l is the pullback of a line in $\mathbb{P}^2$ and e_i are the exceptional divisors associated to each point. Note that the line l' corresponding to the pullback of a line passing through the first two points is a 2-secant line to C_2, i.e., $C\cdot l' = 2$. Consider now the pencil of planes $P \subset \mathbb{P}^3$ containing the line l'. For each such plane P, it follows that $P\cap T = l'\cup Q_P$, where Q_P is a conic curve. Moreover, $10 = P\cdot C_2 = l'\cdot C_2+Q_P\cdot C_2 = 2+Q_P\cdot C_2$ and so $Q_P\cdot C_2 = 8$. Denoting E by the exceptional divisor of p_2 and $\tilde{Q}_P$ the strict transform of Q_P under p_2, we have that $(-K_{X_2})\cdot\tilde{Q}_P = 4H\cdot\tilde{Q}_P-E\cdot\tilde{Q}_P = 0$. □

We can continue by exploring the Sarkisov links initiated by the blowup of curves $C \subset \mathbb{P}^3$ of degree $d < 16$, in particular, exploring the blowup of curves C of type (g,d) in (†), to construct the birational maps of $\mathbb{P}^3$ that realize ι_L and ι_M in (5.4). We leave it for future research. However, since the automorphisms ι_L and ι_M of S are described from the linear projections of $\mathbb{P}^3$ from the lines L and M, respectively, we expect that the desired Cremona transformations contain the Sarkisov link initiated by the blowup along the lines L and M, as the first link in a Sarkisov decomposition.

Chapter 6

Open problems

In this chapter, we mention new research directions that emerged from this thesis.

6.1 Gizatullin's problem for higher Picard rank

Investigating Gizatullin's problem for smooth quartic surfaces $S \subset \mathbb{P}^3$ with Picard rank $\rho(S) \geq 3$ is the natural next step to continue with our work. For quartic surfaces $S \subset \mathbb{P}^3$ with $\rho(S) = 2$, when $\mathrm{Aut}(S)$ comes from $\mathrm{Bir}(\mathbb{P}^3)$, we express the generators of $\mathrm{Aut}(S)$ as Cremona transformations obtained from Sarkisov links and their compositions in Section 5.2. We recall that these Sarkisov links are initiated by the blowup of suitable curves C on S, appearing in (†). In particular, the curves we used are general in the Hilbert scheme $\mathcal{H}_{g,d}$ of smooth irreducible curves in $\mathbb{P}^3$ with genus g and degree d. Conversely, for a general member $[C] \in \mathcal{H}_{g,d}$, the general smooth quartic surface containing C has Picard rank two. When $\rho(S) = 2$ and $r(S) = 20$, one of the curves we used has genus 10 and degree 11. In Section 5.3, we observe that by specializing such a curve of genus 11 and degree 10, we recover Oguiso's example and realize an automorphism of S as the Sarkisov link initiated by the blowup of a such curve. This gives us an "easier" description of the Cremona transformation. Thus, one possible way to explore Gizatullin's problem for higher Picard rank is by deformations of the quartic surfaces with Picard rank two we had studied. Moreover, we are interested in how the automorphism group of the deformation surface is related with the automorphism group of the general one.

6.2 Gizatullin's problem for other K3 surfaces

Just as in the case of quartic surfaces in $\mathbb{P}^3$, it is natural to ask whether the automorphisms of any projective K3 surface are induced by birational maps of an ambient space in which it is embedded.

This question is the focus of an ongoing project in collaboration with Michela Artebani and Alice Garbagnati. We are particularly studying the cases where S is a smooth complete intersection of a quadric and a cubic hypersurfaces in $\mathbb{P}^4$, or a smooth complete intersection of three quadrics in $\mathbb{P}^5$. This leads us to consider two generalizations of Gizatullin's problem.

On the one hand, we investigate whether the automorphisms of S arise from birational maps of a Fano threefold with Picard rank one, such as a quadric or cubic hypersurface in $\mathbb{P}^4$, or the intersection of two quadric hypersurfaces in $\mathbb{P}^5$. When S has Picard rank two, we can apply all the tools developed in sections 4.3, 5.1 and 5.2, particularly leveraging results from [CM13, BL15, Zik23b]. On the other hand, we may also ask which automorphisms of S come from Cremona transformations of the ambient space $\mathbb{P}^4$ or $\mathbb{P}^5$, respectively. Notably, the fact that all regular maps of both smooth quadric and cubic hypersurfaces arise from regular maps of $\mathbb{P}^4$ suggests to address this question by first solving the previous one. Subsequently, we can investigate whether the birational maps of the Fano threefolds come from Cremona maps of the projective space.

6.3 Classification of Sarkisov links

Sarkisov links starting from a Fano threefold Y with Picard rank one are not yet completely classified. The condition on the Picard rank of Y ensures that such links must begin with a divisorial extraction $X \longrightarrow Y$, where X has Picard rank two. Since the Sarkisov link is determined by the two extremal contractions on X, a natural first step in this classification is to study X and the extremal contractions. This approach is explored in [JPR05, JPR11, CM13, BL12, BL15], where divisorial contractions $X \longrightarrow Y$ where X is a weak Fano threefold are classified. In particular, [CM13, BL12] and [BL15] provide a classification of curves in Y such that their blowup X is a weak Fano variety and gives rise to Sarkisov links.

A first result for the cases when X is not weak Fano appears in [Zik23b], which classifies smooth curves in $\mathbb{P}^3$ lying in a smooth cubic surface whose blowup generates Sarkisov links. Continuing along these lines, one might ask about Sarkisov links induced by the blowup of curves in $\mathbb{P}^3$ that are contained in smooth quartics where the resulting blowup is not weak Fano. In Proposition 4.3.11, we prove that if the quartic has Picard rank two, such Sarkisov links do not exist, as the lattice structure of the Picard group of the quartic restricts the possible curves. In the ongoing project mentioned in Section 6.2, we find similar conclusions. The aim of this project is to extend this approach by considering smooth quartic surfaces, and more generally, K3 surfaces with higher Picard ranks, to construct and classify Sarkisov links. Since the existence of antiflips are related with the existence of rational curves on the K3 surface with certain numerical intersection conditions, one can leverage the lattice structure of the Picard group of the surface to exclude some cases.

6.4 Inertia and Decomposition groups

One way to construct interesting subgroups of the Cremona group $\mathrm{Bir}(\mathbb{P}^n)$ is by exploiting these concepts. Given a subvariety $X \subset \mathbb{P}^n$, the *decomposition group* $\mathrm{Dec}(X)$ is the subgroup of $\mathrm{Bir}(\mathbb{P}^n)$ of Cremona transformations stabilizing X. Therefore, every element in $\mathrm{Dec}(X)$ induces a birational self-map of X. The elements of $\mathrm{Dec}(X)$ for which the induced birational map is trivial on X form a subgroup denoted by $\mathrm{In}(X)$ and called the *inertia group*. They fit into an exact sequence

$$0 \longrightarrow \mathrm{In}(X) \longrightarrow \mathrm{Dec}(X) \xrightarrow{r} \mathrm{Bir}(X) .$$

When $X \subset \mathbb{P}^n$ is a hypersurface of degree $n+1$ and mild singularities, the pair $(\mathbb{P}^n, X)$ is a canonical Calabi-Yau pair and we can use the volume preserving Sarkisov program to investigate $\mathrm{Dec}(X)$ and $\mathrm{In}(X)$. Some interesting recent results have been achieved using this method in [ACM23, Duc24, dS24b] and [dS24a]. In this framework, Gizatullin's problem asks to identify the image of r. Thus, another natural continuation of our work is studying the structure of the decomposition and inertia groups in these cases. When $S \subset \mathbb{P}^3$ is a smooth quartic surface with $\rho(S) = 2$ and discriminant $r(S) > 57$ or $r(S) = 52$, $\mathrm{Dec}(S)$ is trivial and so $\mathrm{In}(S)$ is. In the remaining cases for which $\mathrm{Aut}(S) \neq \{1\}$, $\mathrm{Dec}(S)$ is not trivial since we realize automorphisms of S as Cremona transformations of $\mathbb{P}^3$. As mentioned before, each such Cremona transformation of $\mathbb{P}^3$ is initiated by blowing up a suitable curve on S. The blowup of linearly equivalent smooth curves on S give different birational maps of $\mathbb{P}^3$ inducing the same automorphism of S, implying that $\mathrm{In}(S)$ is not trivial.

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