The Cremona Group

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1 Introduction

The purpose of this project is to understand the group of the birational automorphisms of n-dimensional projective space by focusing on the study of its generators. This group is called the *n-Cremona group* and is denoted by $Bir(\mathbb{P}^n)$. An element $\varphi \in Bir(\mathbb{P}^n)$ is an invertible rational self map of $\mathbb{P}^n$, i.e.,

$$\varphi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n, \qquad \varphi = (F_0 : ... : F_n),$$

where $F_0, ..., F_n$ are homogeneous polynomials of degree d without common factor. The *degree* of φ is the integer d and it is denoted by $deg(\varphi)$. Every automorphism of $\mathbb{P}^n$ is a birational map, i.e., $Aut(\mathbb{P}^n) \subset Bir(\mathbb{P}^n)$ is a subgroup. Every rational map $\varphi : \mathbb{P}^1 \dashrightarrow \mathbb{P}^1$ can be extended to $\mathbb{P}^1$. Thus $Bir(\mathbb{P}^1) = Aut(\mathbb{P}^1) = PGL_2(\mathbb{C})$ (see example 1). The simplest example of a birational automorphism of $\mathbb{P}^2$ that is not an isomorphism is the so called *standard quadratic transformation* of the plane:

$$\tau : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \qquad (x : y : z) \longmapsto (yz : xz : xy),$$

Indeed, note that $\tau^2(x : y : z) = (x^2yz : xy^2z : xyz^2) = (x : y : z)$ and therefore τ is invertible as rational map (see example 3).

The group of birational automorphisms of an algebraic variety is a birational invariant and therefore an important object of study in birational geometry. Furthermore, more recently it has had applications in discrete and continuous complex dynamics.

The Cremona group was introduced by the Italian mathematician Luigi Cremona in 1863-1865. At the end of the 19th century, Max Noether stated that every birational automorphism of $\mathbb{P}^2$ is a composition of projective linear transformations and the standard quadratic transformation. Therefore,

$$Bir(\mathbb{P}^2) = \langle Aut(\mathbb{P}^2), \tau \rangle. \tag{1}$$

Noether's idea to show this claim was to consider a birational automorphism of $\mathbb{P}^2$, then take a quadratic transformation q satisfying particular properties such that $deg(\phi \circ q) < deg(\phi)$. Thus, by induction, we obtain a map of degree 1, this is a projective linear transformation. However, such quadratic transformation may not exist. The first complete proof is due to Guido Castelnuovo [4]. The strategy in Castelnuovo's proof was based on two decomposition steps. First, he showed that any birational automorphism of $\mathbb{P}^2$ can be factored as a composition of Jonquière maps, i.e., maps which preserve a pencil of lines. Second, he proved that such maps decompose into quadratic maps. As a consequence of Noether-Castelnuovo Theorem and its proof, the Jonquière maps and linear transformations generate the 2-Cremona group.

In higher dimensions the birational geometry of projective varieties becomes more complicated. We do not have an analogue of the Noether-Castelnuovo Theorem. Hilda Hudson [8] and Ivan Pan [11] proved that any set of group generators of the n-Cremona group, $n \geqslant 3$, contains uncountably many transformations of unbounded degree. More recently, J. Blanc, S. Lamy and S.

Zimmermann [3] showed that $Bir(\mathbb{P}^n)$, $n \geqslant 3$, is not generated by $Aut(\mathbb{P}^n)$, the Jonquière maps and any subset that has a smaller cardinality than that of $\mathbb{C}$. In fact, their result is:

Theorem 1. *Fix $n \geqslant 3$. Let $S \subset Bir(\mathbb{P}^n)$ be a subset of elements in the n-Cremona group that has cardinality smaller than that of $\mathbb{C}$, and let $G \subset Bir(\mathbb{P}^n)$ be the subgroup generated by $Aut(\mathbb{P}^n)$, all Jonquière maps and S. Then, there exists a surjective group homomorphism*

$$Bir(\mathbb{P}^n) \longrightarrow\!\!\!\!\rightarrow \mathbb{Z}/2\mathbb{Z},$$

such that G is contained is its kernel. In particular, G is a proper subgroup of $Bir(\mathbb{P}^n)$.

Theorem 1 is the motivation of our work. We are interested in studying its proof and some tools used in [3]. To show this last result, Blanc, Lamy and Zimmermann used a powerful tool coming from the MMP. This is called the *Sarkisov Program*, which provides a decomposition of any birational automorphism of $\mathbb{P}^n$ into elementary links, called *Sarkisov links.*

All the results mentioned so far, as well as the results we will present in this document, are valid for any algebraically closed field of characteristic zero. For simplicity, throughout this project we work over the field $\mathbb{C}$ of complex numbers.

2 Preliminaries

In this short section we want to establish the notation and state some basic results that will allow us to understand the remaining sections. We skip most of the proofs and focus on examples that are important in the development of this work. We refer to [7] and [12] for proofs and more details.

2.1 First definitions and properties

Let us recall some classical notions of algebraic geometry that we will need.

The *n-dimensional complex projective space* $\mathbb{CP}^n$ can be defined as a compactification of $\mathbb{C}^n$ that adds a new point for every direction in $\mathbb{C}^n$, i.e., $\mathbb{CP}^n = \mathbb{C}^n \cup \mathbb{CP}^{n-1}$. Or equivalently

$$\mathbb{CP}^n = \mathbb{P}^n := \mathbb{C}^{n+1}\backslash\{\bar{0}\}\Big/(x_0, ..., x_n) \sim (\lambda x_0, ..., \lambda x_n),\ \lambda \in \mathbb{C}\backslash\{0\}.$$

We denote by $(x_0 : ... : x_n) \in \mathbb{P}^n$ the equivalence class of $(x_0, ..., x_n) \in \mathbb{C}^{n+1}$. We say that $x_0, ..., x_n$ are the *homogeneous coordinates* of $\mathbb{P}^n$. An *algebraic set* $X \subseteq \mathbb{P}^n$ is the locus of points satisfying a set of polynomial equations, i.e.,

$$X = Z(F_1, ..., F_k) := \{(x_1 : ... : x_n) \in \mathbb{P}^n | F_1(x_1, ..., x_n) = \cdots = F_k(x_1, ..., x_n) = 0\},$$

where each $F_i \in \mathbb{C}[x_1, ..., x_n]$ is a homogeneous polynomial. If $F \in \mathbb{C}[x_0, ..., x_n]$ is a homogeneous polynomial, the algebraic set $Z(F)$ is said to be a *hypersurface* defined by F. If F is of degree one, $Z(F)$ is called a *hyperplane.* We say $X \subseteq \mathbb{P}^n$ is *irreducible* if X is not the union of proper algebraic subsets of X, i.e., if any writing $X = X_1 \cup X_2$, where X_1, X_2 are two algebraic sets implies that $X = X_1$ or $X = X_2$. The *Zariski topology* on $\mathbb{P}^n$ is the topology whose closed subsets are the algebraic sets of $\mathbb{P}^n$. We say that X is a *projective algebraic variety* if X is an irreducible algebraic set $\mathbb{P}^n$.

We define the *ring of regular functions* $\mathcal{O}(X)$ as the set of functions $f : X \longrightarrow \mathbb{C}$ such that for any point $x \in X$ there is an open neighborhood $U \subset X$ of x, and two homogeneous polynomials $F, G \in \mathbb{C}[x_0, ..., x_n]$ of the same degree with G not vanishing at any point of U, such that $f = F/Q$ on U. We define the *field of rational functions* $\mathbb{C}(X)$ as a set of equivalence classes of pairs (U, f) where $U \subset X$ is an open set and $f \in \mathcal{O}(U)$. Where two pairs (U, f) and (V, g) are equivalent if $f = g$ on $U \cap V$. We have that $\mathcal{O}(\mathbb{P}^n) = \mathbb{C}(\mathbb{P}) = \mathbb{C}$, while $\mathbb{C}(\mathbb{P}^n) \cong \mathbb{C}(y_1, ..., y_n)$.

Let $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$ be projective varieties. A *morphism* $\varphi : X \longrightarrow Y$ between projective varieties is a map such that for any point $x \in X$, there is an open subset $U \subseteq X$ which contains x, and homogeneous polynomials $F_0, ..., F_m \in \mathbb{C}[x_0, ..., x_n]$ of the same degree such that $\forall y \in U$, $F_i(y) \neq 0$ for some $i \in \{0, .., m\}$ and $\varphi(y) = (F_0(y) : ... : F_m(y))$. When X and Y are projective spaces, any morphism is globally polynomial.

Proposition 1. *Let $\varphi : \mathbb{P}^n \longrightarrow \mathbb{P}^m$ be a morphism of projective varieties. Then, there exist homogeneous polynomials $F_0, ..., F_m \in \mathbb{C}[x_0, ..., x_n]$ of the same degree such that $\forall x \in \mathbb{P}^n$, $\varphi(x) = (F_0(x) : ... : F_m(x))$.*

We say that $\varphi : X \longrightarrow Y$ is an *isomorphism* if there exists a morphism $\psi : Y \longrightarrow X$ such that $\psi \circ \varphi = Id_X$ and $\varphi \circ \psi = Id_Y$.

Example 1 (Automorphisms of $\mathbb{P}^n$)**.** An automorphism $\varphi : \mathbb{P}^n \longrightarrow \mathbb{P}^n$ is given by $\varphi(x) = (F_0(x) : ... : F_n(x))$ where $F_0, ..., F_n \in \mathbb{C}[x_0, ..., x_n]$ are linearly independent homogeneous polynomials of degree 1. We see that $F_i = a_{i0}x_0 + ... + a_{in}x_n$, so we obtain an isomorphism between $Aut(\mathbb{P}^n)$ and $PGL_{n+1}(\mathbb{C})$, where for each ϕ corresponds the equivalence class $(a_{ij})_{0 \leqslant i,j \leqslant n}$.

Remark 1 (Pull-back of rational functions)**.** If $\varphi : X \longrightarrow Y$ is a morphism between projective surfaces and $f \in \mathbb{C}(Y)^*$ is a rational function on Y, the *pull-back of f by φ* defined by $\varphi^* f := f \circ \varphi$ is a rational function on X.

Now, let us define a more general type of map between projective varieties which is important for their classification.

Definition 1 (Rationals Maps)**.** Let $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$ be projective varieties. A *rational map*, denoted by $\varphi : X \dashrightarrow Y$, is a morphism from a non-empty open subset $U \subseteq X$ to Y which cannot be extended to any larger open set. We say that φ is defined at $x \in X$ if $x \in U$. The *indeterminacy set* of φ is defined by $Ind(\varphi) = X \backslash U$.

Remark 2. If $U \subset X$ is an open subset and $\varphi : U \longrightarrow Y$ is a morphism. Then φ can be extended in a unique way to a rational map, i.e., if $\varphi, \psi : X \dashrightarrow Y$ are rational maps that coincide on a non-empty open subset then $\varphi = \psi$.

When $X = Y = \mathbb{P}^n$, rational maps can be described in a simpler way.

Proposition 2 (Rational Maps of $\mathbb{P}^n$)**.** *Let $\varphi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be a rational map. Then, there exist homogeneous polynomials $F_0, ..., F_n \in \mathbb{C}[x_0, ..., x_n]$ of the same degree, without common factors, such that $\varphi(x_0 : ... : x_n) = (F_0(x_0, ..., x_n) : ... : F_n(x_0, ..., x_n))$.*

As a consequence, $Ind(\varphi) = Z(F_0, ..., F_n)$ is a closed subset of $\mathbb{P}^n$ of codimension $\geqslant 2$. Since the polynomials that describe φ are of same degree, we can define the *degree of* φ as being $deg(\varphi) = deg(F_i)$.

In general, a rational map $\varphi : X \dashrightarrow Y$ is called *birational* if there exist non-empty open subsets $U \subseteq \mathbb{P}^n$ and $V \subseteq \mathbb{P}^m$ such that $\varphi|_U : U \xrightarrow{\sim} V$ is an isomorphism. In this case, X and Y are said to be *birationally equivalent* or simply *birational*.

Here are two examples that play an important role in the main results of this text.

Example 2. Every automorphism $\varphi : \mathbb{P}^n \longrightarrow \mathbb{P}^n$ is a birational map, where the open sets are $U = V = \mathbb{P}^n$, $Ind(\varphi) = \varnothing$ and $deg(\varphi) = 1$.

Example 3 (Standard Quadratic Transformation)**.** Consider the complex projective plane $\mathbb{P}^2$ with homogeneous coordinates x, y, z. The rational map $\tau : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ defined by

$$(x : y : z) \dashrightarrow (yz : xz : xy)$$

is called *standard quadratic transformation*. Fixing the points $p_1 = (1 : 0 : 0)$, $p_2 = (0 : 1 : 0)$ and $p_3 = (0 : 0 : 1)$, we can see that τ is defined on the open set $\mathbb{P}^2 \backslash \{p_1, p_2, p_3\}$. Consider the lines $L_1 = Z(x)$, $L_2 = Z(y)$ and $L_3 = Z(z)$, we have that $\tau(L_i) = p_i$. These lines are called the *exceptional lines*.

If we consider the open set $U := \mathbb{P}^2 \backslash Z(xyz)$ and $(x : y : z) \in U$, we can see that $\tau(\tau(x : y : z)) = (x : y : z)$, that is, $\tau|_U : U \longrightarrow U$ is an isomorphism. Therefore, τ is a birational map, with $\tau^{-1} = \tau$, $Ind(\tau) = \{p_1, p_2, p_3\}$ and $deg(\tau) = 2$.

2.2 Divisors

Now, we will study the notion of divisors on projective varieties.

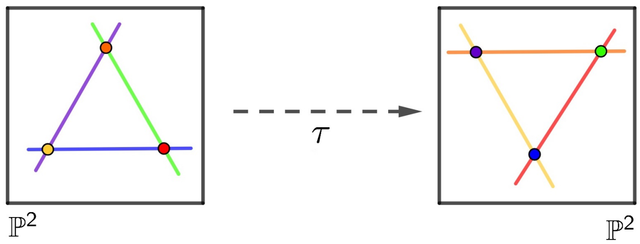

Figure 1: Standard quadratic transformation.

Let X a projective algebraic variety. A *prime divisor* D on X is simply a closed subvariety of X of codimension 1.

Example 4. If X is a curve, the prime divisors are exactly the points of X. If X is a surface, the prime divisors are the irreducible curves that lie on X. If $X = \mathbb{P}^n$, the prime divisors corresponds to irreducible hypersurfaces, i.e., $D = Z(F)$, where $F \in \mathbb{C}[x_0, ..., x_n]$ is an irreducible homogeneous polynomial.

A *Weil divisor* on X is a formal finite sum of prime divisor with integer coefficients. These divisors form an abelian group, denoted by $Div(X)$.

$$Div(X) = \left\{ \sum_{i=1}^{m} a_i D_i | m \in \mathbb{N},\ a_i \in \mathbb{Z},\ D_i \text{ is a prime divisor on } X, \text{ for } i = 1, ..., m \right\}.$$

If $D = \sum a_i D_i$ is a Weil divisor, we call D *effective* if $a_i \geqslant 0$ for every i. Given a rational function $f \in \mathbb{C}(X)^*$, for every prime divisor D we associate the integer $\mathcal{V}_f(D)$ as follows: $\mathcal{V}_f(D) = k > 0$ if f vanishes on D at the order k; $\mathcal{V}_f(D) = -k < 0$ if f has a pole of order k on D, and $\mathcal{V}_f(D) = 0$ otherwise. $\mathcal{V}_f(D)$ is called *multiplicity* of f at D. Since $\mathcal{V}_f(D) = 0$ for all but finitely many D, we define

$$Div(f) = \sum_{D \text{ prime divisor}} \mathcal{V}_f(D)D \quad \in Div(X).$$

The divisors obtained in this way are called *principal divisors.* If $f, g \in \mathbb{C}(X)$ are rational functions, then $div(fg) = div(f) + div(g)$. It follows that principal divisors form a subgroup of $Div(X)$.

Two divisors D, D' on X are *linearly equivalent* is $D - D'$ is a principal divisor. We represent this relation as $D \sim D'$. The quotient of $Div(X)$ by the subgroup of principal divisors is denoted by $Cl(X)$ and is called *divisor class group.*

Example 5 (Divisors of $\mathbb{P}^n$)**.** Any Weil divisor D on $\mathbb{P}^n$ is given by $D = \sum_{i=1}^m a_i Y_i$, where $a_i \in \mathbb{Z}$ and Y_i is an irreducible hypersurface. We define *degree of* D by $degD = \sum a_i deg(Y_i)$, where $deg(Y_i)$ is the degree of the irreducible polynomial that defines Y_i.

We refer [7, Proposition 6.4, Chapter II] to for the proof of the following proposition.

Proposition 3. *Let H be a hypersuperface of $\mathbb{P}^n$. We have the following:*

i) For any $f \in \mathbb{C}(\mathbb{P}^n)^*$, $deg(div(f)) = 0$.

ii) If D is a divisor of degree d in $\mathbb{P}^n$, then $D \sim dH$.

iii) $\theta : Div(\mathbb{P}^n) \longrightarrow \mathbb{Z}$, defined by $\theta(D) = deg(D)$ is a group homomorphism and induces a group isomorphism $\theta : Cl(\mathbb{P}^n) \longrightarrow \mathbb{Z}$.

As consequence, we have that $Cl(\mathbb{P}^n) = \mathbb{Z}[H]$.

2.2.1 Locally principal divisors

Suppose that X is a smooth projective variety and D is a divisor on X. Then, every point $x \in X$ has a neighbourhood in which D is principal. Indeed, for any prime divisor D and any point $x \in X$ there exists an open neighbourhood $U \subset X$ of x in which D is defined by a local equation f, where $f \in \mathbb{C}(X)^*$. So, if $D = \sum a_i D_i$ we can take an open set U with $x \in U$ in which each of the D_i is defined by f_i, then we have $D = div(f)$, where $f = \prod f_i^{a_i}$.

Definition 2 (Pull-back of divisors)**.** Let X and Y be smooth projective varieties and $\varphi : X \longrightarrow Y$ a surjective morphism. Let $D \in Div(Y)$, $D = \sum a_i D_i$. We define the *pull-back of* D $\varphi^* \in Div(X)$ as

$$\varphi^* D := \sum a_i \varphi^* D_i,$$

where, if D_i is locally defined by f_i, $\varphi^* D_i$ is locally defined by $\varphi^* f_i$ (see remark 1).

3 Surfaces

In this section, we want to focus on studying the *plane Cremona group*, $Bir(\mathbb{P}^2)$. We will study the celebrated Noehter-Castelnuovo theorem from a modern point of view. More precisely, using the *Sarkisov program*: a fundamental tool in the current study of Birational Geometry. One of the most important features of this tool is that it can be generalized to higher dimensions. In order to explain the Sarkisov program, we need to introduce some fundamental concepts from intersection theory and birational geometry of projective surfaces.

Throughout this section, we refer to X as a *surface* if it is a smooth projective surface over $\mathbb{C}$. Recall from example 4 that irreducible curves on X are the prime divisors of X.

3.1 Intersection of two curves on a surface

Theorem 2 (Intersection form on surfaces)**.** *Let X be a surface. There exists a unique symmetric bilinear form*

$$\begin{array}{rcl} \cdot : Div(X) \times Div(X) & \longrightarrow & \mathbb{Z} \\ (C, D) & \longmapsto & C \cdot D \end{array}$$

such that:

1) *If C and D are smooth curves on X meeting transversely, then, $C \cdot D = \#(C \cap D)$, the number of points of $C \cap D$.*

2) *If $C \sim C'$, then $C \cdot D = C' \cdot D$ for any $D \in Div(X)$. This means that the intersection number $C \cdot D$ depends only on the linear equivalence classes of C and D.*

For a proof we refer to [7, Theorem 1.1, Chapter V]. By *2)*, $\cdot$ induces an intersection form $Cl(X) \times Cl(X) \longrightarrow \mathbb{Z}$.

Example 6. If $X = \mathbb{P}^2$, the intersection form is given as follows: If C, D are curves of degree n and m respectively, then $C \cdot D = mn$. Indeed, let L be a line in $\mathbb{P}^2$, $L^2 = L \cdot L = 1$. This follows from the fact that two different lines meet transversally at a single point and are linearly equivalent. Since $C \sim nL$ and $D \sim mL$, then $C \cdot D = nL \cdot mL = nmL^2 = nm$.

3.2 Blow-up of a surface at a point

A classical example of morphism which is birational but not an isomorphism is the *blow-up*. The basic idea of blow-ups in algebraic geometry is to remove a point from an algebraic variety and replace it by all the directions pointing out of that point (see figure 2). The blow-up plays a fundamental role in the theory of resolution of singularities and in our case it is an indispensable tool.

Let p be a point in a surface X, we say that $\pi : Y \longrightarrow X$ is the *blow-up of X at p* if:

- Y is a smooth projective surface,
- $\pi|_{Y\backslash\pi^{-1}(p)} : Y\backslash\pi^{-1}(p) \longrightarrow X\backslash\{p\}$ is an isomorphism,
- $\pi^{-1}(p) \cong \mathbb{P}^1$

We call $E := \pi^{-1}(p)$ the *exceptional divisor* or *exceptional curve* of the blow-up. The blow-up of a surface at a point is well defined. In fact, we have the following universal property:

Proposition 4 (Universal property of the blow-up)**.** *Let $\pi : Y \longrightarrow X$ and $\pi' : Y' \longrightarrow X$ be blow-ups of X at $p \in X$. There is a unique isomorphism $\varphi : Y \longrightarrow Y'$ such that $\pi'\varphi = \pi$.*

We refer to [7, Proposition 7.14, Chapter II] for the proof. We denote the blow-up of X at p by $Bl_p(X)$. Given an irreducible curve C passing through p, we define $\tilde{C} := \overline{\pi^{-1}(C\backslash\{p\})} \subseteq Bl_p(X)$. $\tilde{C}$ is called the *strict transform of* C.

Let $\pi : Bl_p(X) \longrightarrow X$ be the blow-up of X at $p \in X$. The birational morphism π induces a group homomorphism

$$\pi^* : Cl(X) \longrightarrow Cl(Bl_p(X)), \qquad C \longmapsto \pi^*(C),$$

(see definition 2).

Now, suppose that X is birational to $\mathbb{P}^2$ (see Definition 6), which will be the case of all the surfaces that we will treat. If $C \subset X$ is a curve and p is a point of x, we can find an affine open neighbourhood (local chart) U of p in X with $U \subseteq \mathbb{C}^2$. Also, we can assume that $p = (0,0)$ and that C is described by the local equation $f = \sum_{i=1}^{n} F_i(x,y) = 0$ in this affine neighborhood, where $F_i's$ are homogeneous polynomials of degree i.

Definition 3 (Multiplicity of a curve at a point)**.** Let $C \subset X$ be a curve on X and let f be a local equation of C at the point p as above. The *multiplicity of* C *at* p, $m_p(C)$, is the lowest i such that F_i is not equal to 0.

We have the following properties:

i) $m_p(C) \geqslant 0$;

ii) $m_p(C) = 0$ if and only if $p \notin C$;

iii) $m_p(C) = 1$ if and only if p is a smooth point of C.

Now, we have a specific description of $\pi^*(C)$ for C a curve on X:

Proposition 5. *Let* $\pi : Bl_p(X) \longrightarrow X$ *be the blow-up of* X *at* $p \in X$. *If* C *is a curve on* X, *then*

$$\pi^*(C) = \tilde{C} + m_p(C)E,$$

where $\tilde{C}$ *is the strict transform if* C *and* $E = \pi^{-1}(p)$ *is the exceptional divisor.*

The following proposition gives a characterization of the divisor class group of the blow-up and describes the intersection form on Bl_pX induced by the intersection form on X.

Proposition 6. *Let* $\pi : Bl_p(X) \longrightarrow X$ *be the blow-up of* X *at* $p \in X$. *Denote by* $E \subset Bl_p(X)$ *the exceptional divisor* $\pi^{-1}(p) \cong \mathbb{P}^1$. *Then,*

$$Cl(Bl_p(X)) = \pi^*(Cl(X)) \oplus \mathbb{Z} \cdot [E].$$

Furthermore, the intersection form on $Bl_p(X)$ *is induced by the intersection form on* X *by the following formulas:*

1) $\pi^*(C) \cdot \pi^*(D) = C \cdot D$ *for any* $C, D \in Cl(X)$,

2) $\pi^*(C) \cdot E = 0$ *for any* $C \in Cl(X)$,

3) $E^2 = E \cdot E = -1$,

For the proofs of previous propositions, we refer to [7, Proposition 3.6, Chapter V] and [7, Proposition 3.2, Chapter V] respectively.

On a surface, curves with properties such as the exceptional divisor have a particular name.

Definition 4 ((-1)-curve)**.** A curve C on a surface X is said to be a (-1)-*curve* if $C^2 = -1$ and $C \cong \mathbb{P}^1$.

From Proposition 6, the exceptional divisor of a blow-up of a surface at a point is a (-1)-curve. A natural question is: Is any (-1)-curve in a surface Y the exceptional divisor of some blow-up of a surface at a point? The answer is:

Proposition 7 (Castelnuovo's contractibility criterion)**.** *Let $C \subset Y$ be a (-1)-curve in a surface Y. Then, there exists a surface X, a point $p \in X$ and a morphism $\pi : Y \longrightarrow X$, such that $Y \backslash C \cong X \backslash \{p\}$ via π, and C is the exceptional divisor.*

For a proof we refer to [7, Theorem 5.7, Chapter V].

3.2.1 Blow-up of $\mathbb{P}^2$ at a point

In order to understand the blow-up construction of the blow-up, we restrict our focus to the smooth surface $X = \mathbb{P}^2$.

For a point $p \in \mathbb{P}^2$, say $p = (0:0:1)$. Consider the projection from the point p, $\pi_p : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ given by $(x:y:z) \longmapsto (x:y)$. This is a rational map defined in $\mathbb{P}^2 \backslash \{p\}$. Let $\Gamma_{\pi_p} = \{((x:y:z),(x:y)) | (x:y:z) \neq p\} \subset \mathbb{P}^2 \times \mathbb{P}^1$ be the graph of π_p. Define $X := \overline{\Gamma}_{\pi_p}$ the closure of Γ_{π_p} in $\mathbb{P}^2 \times \mathbb{P}^1$ (see figure 2).

Proposition 8. (Blow-up of $\mathbb{P}^2$ at p)*. The map $\pi : X \longrightarrow \mathbb{P}^2$ is the blow-up of $\mathbb{P}^2$ at p, where π is the projection on the first factor.*

Indeed, let $((x:y:z),(t:s))$ be homogeneous coordinates for $\mathbb{P}^2 \times \mathbb{P}^1$. We can see that $X = Z(xs - yt) \subset \mathbb{P}^2 \times \mathbb{P}^1$, then $E = \pi^{-1}(p) = \{p\} \times \mathbb{P}^1 \cong \mathbb{P}^1$ is the *exceptional divisor* of the blow-up, and π restricts to an isomorphism $\pi|_{X \backslash E} : X \backslash E \xrightarrow{\sim} \mathbb{P}^2 \backslash \{p\}$.

Let C be a curve passing through p, $\tilde{C} = \overline{\pi^{-1}(C \backslash \{p\})}$ is the *strict transform of C* (see figure 2).

Theorem 3 (Resolution of Indeterminacy)**.** *Let $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a birational map. Then there exists a surface X and a morphism $\psi : X \longrightarrow \mathbb{P}^2$ which is the composition of a finite number of blow-ups such that the composition $\varphi \circ \psi$ is a morphism.*

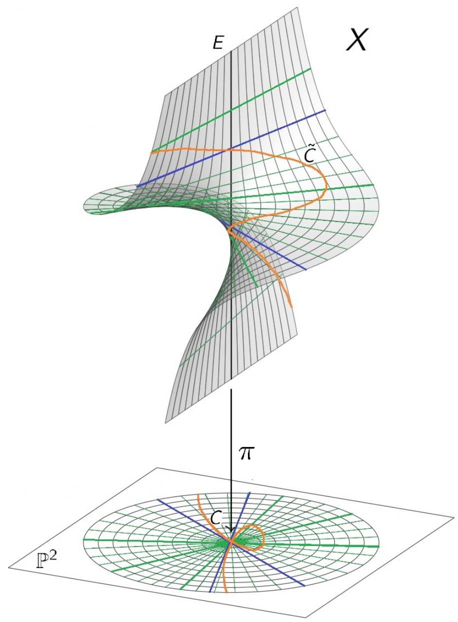

Figure 2: Blow-up of $\mathbb{P}^2$ at the point p.

We refer to [12, Theorem 3, Chapter IV] for the proof of the previous theorem but let us give an idea. Given a birational map $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, the indeterminacy set $Ind(\varphi)$ is finite. If there exists $p_1 \in Ind(\varphi)$, denote by $\pi_1 : X_1 \longrightarrow \mathbb{P}^2$ the blow-up at this point. The map $\varphi_1 = \varphi \circ \pi_1$ is a birational map $\varphi_1 : X_1 \dashrightarrow \mathbb{P}^2$. If $Ind(\varphi_1) \neq \varnothing$, take p_2 a indeterminacy point of φ_1 denoted by $\pi_2 : X_2 \longrightarrow \mathbb{P}^2$ its blow-up. Again, the map $\varphi_2 = \varphi_1 \circ \pi_1$ is a birational map $\varphi_2 : X_2 \dashrightarrow \mathbb{P}^2$. We iterate this process until φ_t becomes a morphism. Such t exists.

$$\begin{tikzcd}
X_t \arrow[d, "\pi_t"'] \arrow[dddr, "\varphi_t"] & \\
\vdots \arrow[d] & \\
X_1 \arrow[d, "\pi_1"'] & \\
\mathbb{P}^2 \arrow[r, dashed, "\varphi"'] & \mathbb{P}^2
\end{tikzcd}$$

Example 7. Consider the standard quadratic transformation τ. By example 3, the indeterminacy points are $(1:0:0), (0:1:0)$ and $(0:0:1)$. Applying the previous process we have the following figure

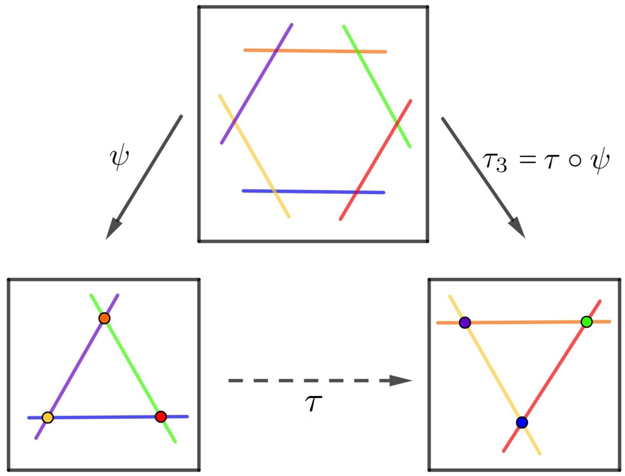

Figure 3: Resolution of indeterminacy of τ.

3.3 Ruled surfaces

Let us now focus our attention on ruled surfaces. We will see how to construct the Hirzebruch surfaces $\mathbb{F}_n$ and their intersection forms. These surfaces play a fundamental role in the Sarkisov program.

Definition 5 (Ruled surfaces)**.** Let X be a surface. We say that X is a *scroll* if there is a surjective morphism $\pi : X \longrightarrow B$ onto a smooth curve B whose fibers are all isomorphic to $\mathbb{P}^1$. A surface birationally equivalent to a scroll is called a *ruled surface.*

Definition 6 (Rational surfaces)**.** We say that a surface X is *rational* if there is a birational map $\phi : X \dashrightarrow \mathbb{P}^2$.

Example 8. $\mathbb{P}^1 \times \mathbb{P}^1$ is a rational surface. In fact, let $((t:s),(u:v))$ and $(x:y:z)$ be the homogeneous coordinates of $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ respectively. Consider the open sets $U = (\mathbb{P}^1 \backslash Z(t)) \times (\mathbb{P}^1 \backslash Z(u)) \subset \mathbb{P}^1 \times \mathbb{P}^1$, $V = \mathbb{P}^2 \backslash Z(x)$ and the isomorphism $\varphi : U \longrightarrow V$ defined by $((1:s),(1:v) \longmapsto (1:s:v)$. Then, φ can be extended in a unique way to a birational map from $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^2$.

Example 9 (Construction of Hirzebruch surfaces)**.** Consider the surface $\mathbb{P}^1 \times \mathbb{P}^1$ and $m \in \mathbb{N}$. Let us take the m lines $L_i = (i:1) \times \mathbb{P}^1$, $1 \leqslant i \leqslant m$, and $L = \mathbb{P}^1 \times (0:1)$. Let $p_1, ..., p_m \in \mathbb{P}^1 \times \mathbb{P}^1$ be the intersection points of the lines $L_1, ..., L_m$ with L respectively. Blowing up the surface at these m points we get a surface with $E_1, ..., E_m$ and $\tilde{L}_1, ..., \tilde{L}_m$ (-1)-curves, where E_i is the exceptional curve of blow-up of p_i in $\mathbb{P}^1 \times \mathbb{P}^1$, $\tilde{L}_i$ is the strict transform of the line L_i, and $\tilde{L}_i \cap \tilde{L}_j = \varnothing$ for $1 \leqslant i \neq j \leqslant m$. By Castelnuovo's criterion (proposition 7), we can contract every curve $\tilde{L}_i$ to a point. The surface we get is called *Hirzebruch surface* $\mathbb{F}_m$. After these blow-ups and blow-downs, we have a birational map $\varphi_m : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{F}_m$. This surface has the structure of a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ $\pi : \mathbb{F}_m \longrightarrow \mathbb{P}^1$. Denote by f a fiber of π and by $E = \varphi(L) \cong \mathbb{P}^1$, if $m \geqslant 1$, E is the unique section such that $E^2 = -m$. In fact, for any section E' of π different from E, we have that $E'^2 \geqslant m$. By Proposition 6 we have

$$Cl(\mathbb{F}_m) = \mathbb{Z}[f] \oplus \mathbb{Z}[E]$$

and the intersection form on $\mathbb{F}_m$ is given by:

- $f^2 = 0$;
- $f \cdot E = 1$;
- $E^2 = -m$.

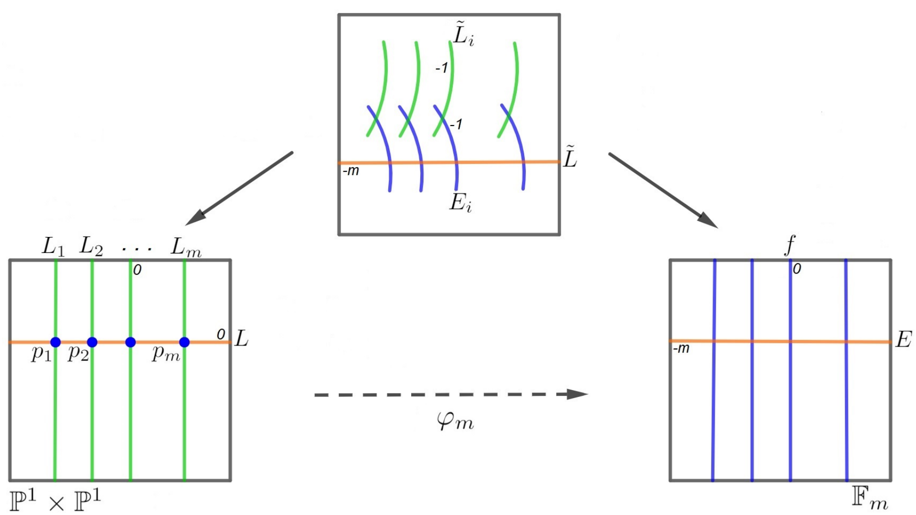

Figure 4: Construction of Hirzebruch surfaces.

We write $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$. For $n = 1$ we can see that $\mathbb{F}_1$ is isomorphic to the blow-up of $\mathbb{P}^2$ at one point.

Definition 7 (Rational ruled surfaces)**.** Let X be a surface. We say that X is a *rational ruled surface* if there is a surjective morphism $\pi : X \longrightarrow \mathbb{P}^1$ such that every fiber of π is isomorphic to $\mathbb{P}^1$.

Example 10. Hizerbruch surfaces are rational ruled surfaces. If we have that $\mathbb{F}_m \cong \mathbb{F}_n$, *then* $m = n$.

The following result states that the only rational ruled surfaces are the Hizerbruch surfaces (up to isomorphism).

Proposition 9. *Let X be a rational ruled surface. Then, there exists $n \geqslant 0$ such that $X \xrightarrow{\sim} \mathbb{F}_n$.*

Definition 8 (Minimal rational surfaces)**.** $\mathbb{P}^2$ together with Hizerbruch surfaces can be considered the simplest surfaces in the birational class of rational surfaces and are called *minimal rational surfaces* (see Section 4).

3.4 Birational automorphisms of the plane

Now, let us study the birational self maps of $\mathbb{P}^2$. Since $\mathbb{P}^2$ is a smooth surface, we can apply previous results. Let $(x : y : z)$ be homogeneous coordinates for $\mathbb{P}^2$.

Definition 9 (The Cremona group of the plane)**.** The group of all birational maps $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is called *the Cremona group of the plane*, and is denoted by $Bir(\mathbb{P}^2)$. An element $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ in $Bir(\mathbb{P}^2)$ is called *birational transformation of* $\mathbb{P}^2$ or *Cremona transformation* and is given by

$$(x : y : z) \longmapsto (F_0(x, y, z) : F_1(x, y, z) : F_2(x, y, z)),$$

for some homogeneous polynomials $F_0, F_1, F_2 \in \mathbb{C}[x, y, z]$ of the same degree d, without common factors. We defined the *degree of* ϕ by $deg(\phi) := d$ and the *indeterminacy set of* ϕ by $Ind(\phi) := Z(F_0, F_1, F_2)$.

As we saw in the previous section, $Aut(\mathbb{P}^2)$ is a subgroup of $Bir(\mathbb{P}^2)$. Any element of $Aut(\mathbb{P}^2)$ is a Cremona map of degree 1. Let us recall that the standard quadratic transformation $\tau : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is defined by

$$(x : y : z) \longmapsto (yz : xz : xy),$$

having degree 2. The following is the celebrated theorem of Max Noether and Guido Castelnuovo:

Theorem 4 (Noether-Castelnuevo Theorem)**.** *The group $Bir(\mathbb{P}^2)$ is generated by $Aut(\mathbb{P}^2)$ and the standard quadratic transformation. That is*

$$Bir(\mathbb{P}^2) = \langle Aut(\mathbb{P}^2), \tau \rangle.$$

As we mentioned in the introduction, this theorem was proved by Castelnuovo using particular Cremona maps.

Definition 10 (Jonquère maps)**.** Let $J : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a birational self map of $\mathbb{P}^2$. J is said to be a *Jonquière map* if there exists $p, q \in \mathbb{P}^2$ such that J takes all lines going through p to lines going through q (up to a finite number of lines).

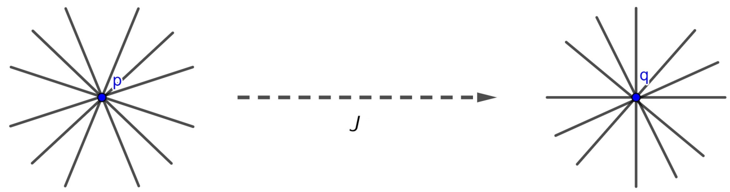

Figure 5: Jonquière map.

Example 11. The standard quadratic transformation is a Jonquière map. In fact, take $p = q = (0:0:1)$ and L any line in $\mathbb{P}^2$ passing through p. If L is different from $Z(x)$ and $Z(y)$, $J(L)$ is a line going through p.

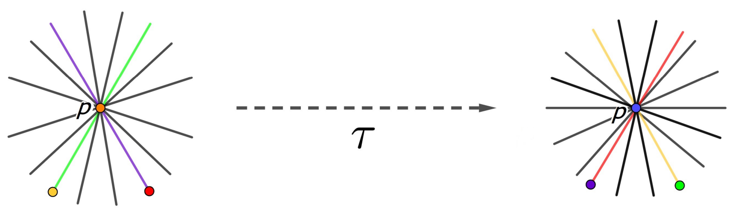

Figure 6: The standard quadratic transformation is a Jonquière map.

The strategy of proof of Nother-Castelnuovo Theorem given by castelnuovo is based on two decomposition steps. The first step is:

Theorem 5. *Every birational self map of* $\mathbb{P}^2$ *can be factored as a composition of Jonquière maps.*

The second step is:

Theorem 6. *Every Jonquière map is written as composition of automorphisms of* $\mathbb{P}^2$ *and* τ*.*

3.5 Sarkisov program of the plane

J. Kollár, K. Smith and A. Corti gave another proof of Theorem 4. Their proof follows the general outline of Castelnuovo's proof, with a modern point of view, with a perspective from the Sarkisov program (see [10, Section 2.5]).

Definition 11 (Sarkisov program)**.** The *Sarkisov program* is an algorithm for decomposing birational maps between minimal rational surfaces into elementary links.

Before giving a description of the program, let us define such elementary links.

3.5.1 Sarkisov links

There are four types of elementary links:

I The inverse $\chi_1 : \mathbb{P}^2 \dashrightarrow \mathbb{F}_1$ of a point blow-up.

II The *elementary transformation* $\chi_2 : \mathbb{F}_m \dashrightarrow \mathbb{F}_{m\pm1}$, defined as the blowing up of a point $p \in \mathbb{F}_m$, followed by the contraction of the birational transform of the fiber through p (see figures 7 and 8).

III The blow-up $\chi_3 : \mathbb{F}_1 \longrightarrow \mathbb{P}^2$ of a point $p \in \mathbb{P}^2$.

IV The morphism $\chi_4 : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ which exchanges the two factors.

We have two cases in the link of type II:

II.1 If $p \in \mathbb{F}_m$ belongs to the negative section E of $\mathbb{F}_m$ (or $m = 0$), the elementary link is $\chi_2 : \mathbb{F}_m \dashrightarrow \mathbb{F}_{m+1}$. In fact, let $\pi : Bl_p(\mathbb{F}_m) \longrightarrow \mathbb{F}_m$ be the blow-up of the point p and let F be its exceptional divisor. By Proposition 6, the strict transform $\tilde{E}$ of the negative section E has self-intersection $\tilde{E}^2 = m-1$. Let f be the fiber through p and $\tilde{f}$ its strict transform. Then $\tilde{f}$ is a (-1)-curve in $Bl_p(\mathbb{F}_m)$ and we have that $\tilde{E}\cdot\tilde{f} = 0$. From Proposition 7, there exists $\pi' : Bl_p(\mathbb{F}_m) \longrightarrow Y$ where Y is a rational ruled surface, which contracts $\tilde{f}$ to a point q. Thus, $Bl_p(\mathbb{F}_m) = Bl_q(Y)$ and we have that $E' := \pi'(\tilde{E})$ is a section of $Y \longrightarrow \mathbb{P}^1$ with self-intersection $-m-1$. By proposition 9, $Y \cong \mathbb{F}_n$ for some integer $n \geqslant 0$. Therefore, $Y \cong \mathbb{F}_{m+1}$ because Y has a unique curve with negative self-intersection.

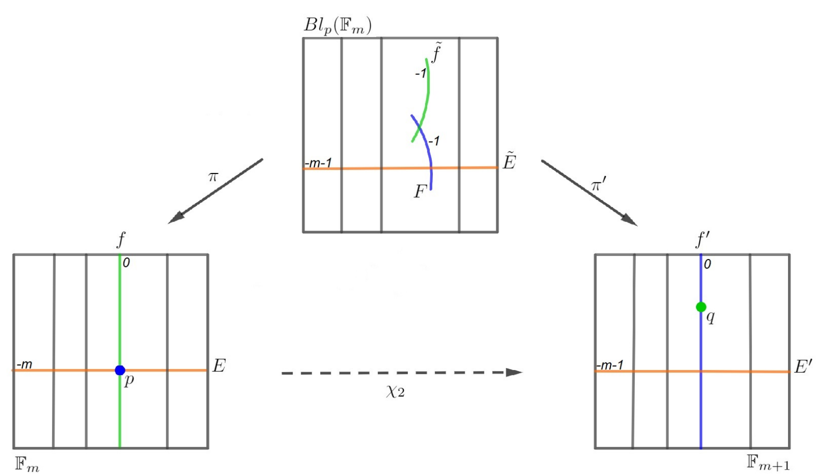

Figure 7: Sarkisov link of type II with $p \in E$.

II.2 If $m \geqslant 1$ and $p \in \mathbb{F}_m$ does not belong to the negative section E of $\mathbb{F}_m$, the elementary link is $\chi_2 : \mathbb{F}_m \dashrightarrow \mathbb{F}_{m-1}$. By a similar argument as above we have that $\tilde{E}^2 = -m$ and $\tilde{E} \cdot \tilde{f} = 1$ in $Bl_p(\mathbb{F}_m)$. So, contracting $\tilde{f}$, $E' = \pi'(\tilde{E})$ has self-intersection $-m+1$ and we get the $\mathbb{F}_{m-1}$ surface.

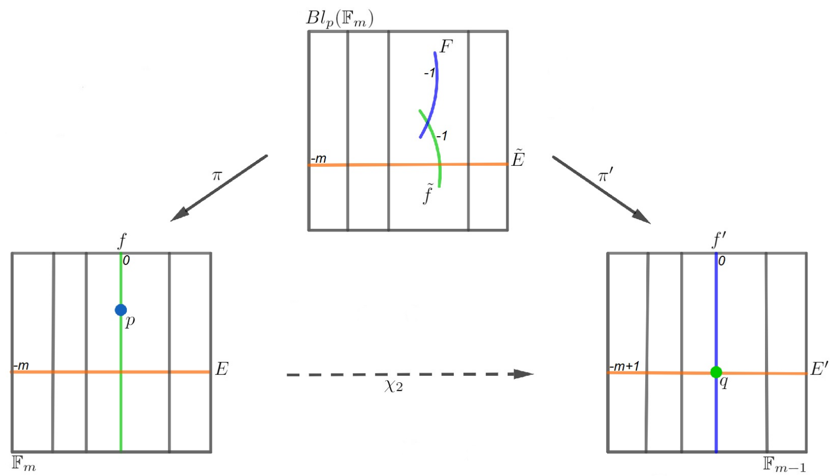

Figure 8: Sarkisov link of type (II) with $p \notin E$.

Note that the elementary links are birational maps between minimal rational surfaces.

Example 12. The map $\varphi_m : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{F}_m$ in example 9 is factored by m elementary links of type (II) 3.5.1. Indeed, take the lines $L_1, ..., L_m, L$ and the points $p_1, ..., p_m$ as in 9. By blowing up of point p_1 and then contracting the strict transform of L_1, we get the surface $\mathbb{F}_1$. By abuse of notation we use the same symbols $L_2, ..., L_m$ and L to denote the curves in $\mathbb{P}^1 \times \mathbb{P}^1$ and their images in $\mathbb{F}_1$, and likewise we use $p_2, ..., p_m$ to denote the intersections of $L_2, ..., L_m$ with L. Now, we perform the blow-up of p_2 and then we contract the strict transform of L_2. Thus, we get the surface $\mathbb{F}_2$. We iterate this process until we have blown-up the point p_m and we obtain the $\mathbb{F}_m$ surface.

Now, we will describe the Sarkisov program.

Theorem 7 (Sarkisov program for surfaces)**.** *Every birational transformation between minimal rational surfaces is a composition of Sarkisov links.*

Given a map $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, let us describe the algorithm for constructing a factorization in the Sarkisov program. For this, we need to consider more generally birational maps from $\mathbb{P}^2$ or $\mathbb{F}_m$, $m \geqslant 0$, to $\mathbb{P}^2$.

Given a map $\psi : X \dashrightarrow \mathbb{P}^2$, where $X = \mathbb{P}^2$ or $X = \mathbb{F}_m$, for $m \geqslant 0$. We find a elementary link $\psi_1 : X \dashrightarrow X_1$, with $X_1 = \mathbb{P}^2$ or $X_1 = \mathbb{F}'_m$, such that $\beta_1 := \psi \circ \psi_1^{-1}$ is "simpler" than ψ. Again, we proceed to find an elementary Sarkisov link $\psi_2 : X_1 \dashrightarrow X_2$ such that $\beta_2 := \beta_1 \circ \psi_2^{-1}$ is simpler than β_1. We continue in this manner and then the Sarkisov program will give us the following factorization for $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$

$$\begin{tikzcd}
\mathbb{P}^2 \arrow[r, dashed, "\psi_1"] \arrow[rrrr, dashed, bend right, "\varphi"'] & X_1 \arrow[r, dashed, "\psi_2"] & X_2 \arrow[r, dashed, "\psi_3"] & \cdots \arrow[r, dashed, "\psi_n"] & X_n = \mathbb{P}^2
\end{tikzcd}$$

Any birational map from a minimal rational surface to $\mathbb{P}^2$ has a corresponding *Sarkisov degree*. This is a numerical invariant that measures the complexity of the map. For example, if $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is defined by homogeneous polynomials F_0, F_1, F_2 of the same degree d, the Sarkisov degree, denoted by $s.deg(\varphi)$ is d. In the previous algorithm, by "β_1 is simpler than ψ", we mean that $s.deg(\beta_1) \leqslant s.deg(\psi)$. Therefore, such factorization into Sarkisov links is done by induction on the Sarkisov degree.

4 Minimal Model Program

The Minimal Model Program (MMP) is a program for the construction a "simplest" representatives of each birational class of projective varieties. In this section, we will give a description of the Minimal Model Program for surfaces and at the end we will discuss the generalizations to higher dimensions. Before describing the MMP for surfaces, let us give some important definitions.

Let X be a smooth projective surface. From the intersection form in Theorem 2, we obtain another equivalence relation in $Div(X)$: let $D, D' \in Div(X)$ be divisors on X, they are said to be *numerically equivalent* if $D \cdot C = D' \cdot C$ for every curve $C \subset X$ and we denote this by $D \equiv D'$. The quotient group of $Div(X)$ by the equivalence relation $\equiv$ is denoted by $Num(X)$ and the Neron-Severi Theorem asserts that this abelian group is finitely generated. Its rank is denoted by $\rho(X)$ and is called the *Picard number of X*.

We define the $\rho(X)$-dimensional $\mathbb{R}$-vector space $N^1(X) := Num(X) \otimes_{\mathbb{Z}} \mathbb{R}$. The intersection form on X induces a nondegenerate symmetric bilinear form $\cdot : N^1(X) \times N^1(X) \longrightarrow \mathbb{R}$.

Example 13. Let $\pi : Bl_p(X) \longrightarrow X$ be the blow-up of X at $p \in X$. Denote by $E \subset Bl_p(X)$ the exceptional divisor $\pi^{-1}(p) \cong \mathbb{P}^1$. Similar to Theorem 6, we have that $N^1(Bl_p(X)) = \pi^*(N^1(X)) \bigoplus \mathbb{R} \cdot [E]$ and thus $\rho(Bl_p(X)) = \rho(X) + 1$.

In the case of surfaces, if X, Y are smooth projective surfaces, by "X is simpler than Y" we mean that $\rho(X) < \rho(Y)$. Note that if $Bl_p(X)$ is the blow-up of a smooth projective surface X at point $p \in X$, $Bl_p(X)$ is a smooth projective surface and by the previous example, X is simpler than $Bl_p(X)$. Now, let us give a notion of a "simplest model" in each birational class of surfaces.

Definition 12 (Minimal surface)**.** A smooth projective surface X is called a *minimal surface* if for every birational morphism $\pi : X \longrightarrow Y$ onto another smooth projective surface Y, we have that π is a isomorphism.

Similar to Theorem 3 we have the following result:

Proposition 10 (Factorization of birational morphism of surfaces)**.** *Let $\pi : X \longrightarrow Y$ be a birational morphism between smooth surfaces. Then π is a composition of a finite number of blow-ups.*

So, every smooth surface can be obtained from a minimal surface in its birational class by a sequence of blow-ups, in fact, by a finite sequence of blow-ups because the Picard number increases by one in each blow-up. Now, given any smooth surface X, we need to do the inverse operation of blow-up to find its minimal surface. Thus, from the Castelnuono's contractibility criterion (Proposition 7) we have a algorithm for the MMP for surfaces.

4.1 Classical version of the MMP for surfaces

1) Start with a smooth projective surface X with Picard number $\rho = \rho(X) \geqslant 0$.

2) If X does not contain a (-1)-curve, X is not the blowup of any smooth surface and X is a minimal surface, stop.

3) If X contains a (-1)-curve, we use the Castelnuovo's contractibility criterion for blow it down. So, we get a birational morphism $\pi : X \longrightarrow X'$ with X' a smooth surface and $\rho(X') = \rho(X) - 1$.

4) Replace X by X' and return to step 1).

Since the Picard number decreases by one in each step, this procedure ends after a finite number of repetitions.

In the birational class of the rational surfaces, the minimal surfaces are exactly the minimal rational surfaces: $\mathbb{P}^2$ and the $\mathbb{F}_m's$ $(m \neq 1)$. So, the minimal surface in a birational class may not be unique.

Example 14 (Blow-up of $\mathbb{P}^2$ at two points)**.** Let X be the blow-up of $\mathbb{P}^2$ at two points p and q. Consider L the line passing through those points. Note that X has three (-1)-curves. The exceptional divisors E_p and E_q of blow-up of p and q respectively, and the strict transform of L, $\tilde{L}$. If we run the MMP for X, we have three options of (-1)-curves to contract. If we first contract the E_p curve (or E_q), the output of the program is the surface $\mathbb{P}^2$. But if we start by contracting the $\tilde{L}$ curve, we get from the MMP the surface $\mathbb{F}_0$.

Note that the definition of a (-1)-curve uses the fact that the ambient variety is a surface. So, the classical version of the MMP cannot be generalized to higher dimensions easily.

4.2 Introduction to the modern version of MMP for surfaces

Let us define some objects needed for the MMP (modern version).

Let X be a smooth surface, we can consider local algebraic coordinates (or local analytic complex coordinates) x_1, x_2. Take $f_1, f_2 \in \mathbb{C}(X)$ such that $\mathbb{C}(f_1, f_2) \subset \mathbb{C}(X)$ is a finite algebraic extension. Given any $g \in \mathbb{C}(X)$, $g \neq 0$, we write formally $s = g \cdot df_1 \wedge df_2$ and call it a *rational 2-form*. We can compare this rational 2-form to the volume element $dx_1 \wedge dx_2$ as follows:

$$s = g \cdot df_1 \wedge df_2 = Jg \cdot dx_1 \wedge dx_2,$$

where $J = det\left|\frac{\partial f_i}{\partial x_j}\right|$. Note that zeros and poles of J are well defined. So, we can get a divisor from s. Given a prime divisor D on X, define $\mathcal{V}_s(D) := \mathcal{V}_{Jg}(D)$ and

$$div(s) = \sum_{D \text{ prime divisor}} \mathcal{V}_{Jg}(D)D \in Div(X).$$

This divisor is a particular and important divisor on a surface X. Its divisor class is frequently used in classification of surfaces.

Definition 13 (Canonical divisor of a surface)**.** The *canonical divisor* or *canonical class* of X is the divisor class $K_X = div(s)$ where s is a 2-form.

It is a well-defined divisor class because two 2-form s, s' are related by $s = hs'$ where $h \in \mathbb{C}(X)^*$ and we have that $div(s) = div(s') + div(h)$.

Example 15 (Canonical divisor for $\mathbb{P}^2$)**.** Consider $\mathbb{P}^2$ with homogeneous coordinates u, v, w. Let $U_w = (w \neq 0) \cong \mathbb{A}^2$ the open subset of $\mathbb{P}^2$ with local coordinates x, y. Take $s = dx \wedge dy$. Now, if we look at the open subset $U_u = (u \neq 0) \cong \mathbb{A}^2$ with local coordinates s, t; the coordinate change is $x = 1/t \in \mathbb{C}(U_u) \cong \mathbb{C}(\mathbb{P}^2)$ and $y = s/t \in \mathbb{C}(U_u) \cong \mathbb{C}(\mathbb{P}^2)$. We have that

$$J = det\begin{pmatrix} \frac{dx}{ds} & \frac{dx}{dt} \\ \frac{dy}{ds} & \frac{dy}{dt} \end{pmatrix} = \frac{1}{t^3}.$$

Then $s = dx \wedge dy = \frac{1}{t^3} ds \wedge dt$ and $K_{\mathbb{P}^2} = div(s) = -3H$, where H is a hyperplane class.

Now, we will reformulate the MMP in terms of the canonical divisor. We start by giving the following concepts.

Definition 14. Let $D \in Div(X)$ be a divisor on a smooth surface X. We say that D is *nef* if $D \cdot C \geqslant 0$ for every curve $C \subset X$.

Remark 3. Let D be an effective divisor on a smooth surface X. *The arithmetic genus* $p_a(D)$ of D is defined by

$$p_a(D) = 1 + \frac{1}{2}(K_X + D) \cdot D.$$

In particular, if C is a curve, $p_a(C)$ coincides with the dimension of the space of rational 1-form. So, $p_a(C) \geqslant 0$. We also have that $p_a(C) = 0$ if and only if $C \cong \mathbb{P}^1$. You can see [7, Section 1, Chapter IV] for more details.

The new notion of a "simplest model" will not be equivalent to the previous one in the case of rational and ruled surfaces. But this notion can be generalized in higher dimensions.

Definition 15 (Minimal model)**.** We say that a smooth projective surface X is a *minimal model* if K_X is nef.

Remark 4. The canonical $K_{\mathbb{P}^2}$ is not nef. In fact, $K_{\mathbb{P}^2} \cdot H = -3H^2 = -3 < 0$. Thus, the minimal surfaces are not necessarily minimal models. From remark 3 we have the following result and it implies that every minimal model is a minimal surface.

Proposition 11. *Let C be a curve on a smooth surface X. Then C is a (-1)-curve if and only if $K_X \cdot C < 0$ and $C^2 < 0$.*

We consider again $N^1(X)$, the $\mathbb{R}$-vector space of divisors of a smooth surface X, or equivalently the space of curves. A subset N of any $\mathbb{R}$-vector space V is called a *cone* if $0 \in N$ and N is closed under multiplication by positive scalars. A subcone $M \subset N$ is called *extremal face* if $\forall u, v \in N$ with $u + v \in M$, then $u, v \in M$. If M is an extremal face with dimension 1, we say that M is an *extremal ray.*

Definition 16 (Mori cone)**.** Let X be a smooth projective surface. Set

$$NE(X) = \left\{ \sum a_i [C_i] | C_i \subset X \text{ is a curve, } 0 \leqslant a_i \in \mathbb{R} \right\} \subset N^1(X),$$

where $[C_i]$ is the numerical class of C_i. It is a cone in $N^1(X)$ and its closure $\overline{NE}(X)$ is called the *Mori cone.*

Any divisor $D \in Div(X)$ defines a linear function $D : N^1(X) \longrightarrow \mathbb{R}$ given by $[C] \longmapsto D \cdot C$ where C is a curve on X (recall that $N^1(X)$ is generated by the classes of the curves). We define $\overline{NE}(X)_{\geqslant 0} := \{x \in \overline{NE}(X) | D \cdot x \geqslant 0\}$ and similarly $\overline{NE}(X)_{=0}$ and $\overline{NE}(X)_{<0}$. If an extremal face $M \subset \overline{NE}(X)$ such that $M \backslash \{0\} \subset \overline{NE}(X) < 0$, we say that M is a *D-negative extremal face.*

The first step to the MMP is to contract some K_X-negative extremal ray. This is an analogue of contracting (-1)-curve in the classical version.

Definition 17. Let X be a smooth surface and R an extremal ray of $\overline{NE}(X)$. A morphism $\varphi_R : X \longrightarrow Y$ onto a normal projective variety Y with connected fiber is a *contraction* of R if the following hold:

$$\varphi_R(C) \text{ for an irreducible curve } C \text{ is a point if and only if } [C] \in R.$$

The next theorem is the analogue to the Castelnuovo's contractibility criterion (Proposition 7) and it asserts that the contraction of any K_X-negative extremal ray always exists. You can refer to [9, Theorem 1.28] for the proof.

Theorem 8. *Let X be a smooth projective surface and $R \subset \overline{NE}(X)$ a K_X-negative extremal. Then $R = \mathbb{R}_{\geqslant 0}[C]$ for some curve $C \subset X$ with $K_X \cdot C < 0$. Furthermore, the contraction φ_R of R exists and is one of the following type:*

1) *If $C^2 < 0$, then $\varphi_R : X \longrightarrow Y$ is the blow-u p of a smooth surface Y at one point; $\rho(Y) = \rho(X) - 1$.*

2) *If $C^2 = 0$, then $\varphi_R : X \longrightarrow Y$ realizes X as a scroll (see definition 5) over a smooth curve Y. C is a fiber of φ_R and $\rho(X) = 2$.*

3) *If $C^2 > 0$, then $X \cong \mathbb{P}^2$ and $\varphi_R : X \longrightarrow pt$; $\rho(X) = 1$.*

Definition 18. A morphism of the type 2) or type 3) in the previous theorem is called *Mori fiber space*.

We already have the tools for describing the MMP.

1) Start with a smooth projective surface X with $\rho = \rho(X)$.

2) If K_X is nef, X is a minimal model, stop.

3) If K_X is not nef, pick a K_X-negative extremal ray $R \subset \overline{NE}(X)$ and apply Theorem 8.

4) If $\varphi_R : X \longrightarrow Y$ is the contraction with $dim(Y) < 2$, φ_R is a Mori fiber space, stop.

5) If $\varphi_R : X \longrightarrow Y$ is the contraction with $dimX = 2$, X is the blow-up of a smooth surface Y at one point and $\rho(Y) = \rho(X) - 1$.

6) Replace X by Y and return to step 1).

4.3 The MMP in higher dimensions

There is an intersection theory in higher dimensions. Let X be a smooth projective variety of dimension n. We have an intersection product resulting from intersecting divisors and curves on X, where a divisor D on X is a formal sum of irreducible closed subsets of X of codimension 1 (see 2.2). It is important to mention that divisors and curves coincide only on surfaces. From *rational n-forms* (defined in the same way as section 4.2) we obtain the *canonical divisor* K_X and it is *nef* if $D \cdot C \geqslant 0$ for every curve $C \subset X$. The MMP (modern version) can be extended to any dimension. Given a smooth projective variety X, the first step is to ask if K_X is nef. If K_X is nef, X is a minimal model and we stop. If K_X is not nef, the task is to find a K_X-negative extremal ray R which can be contracted. If $\varphi_R : X \longrightarrow Y$ is its contraction we have two possibilities:

(a) $dimX < dimY$, then we stop the program and call $\varphi_R : X \longrightarrow Y$ a *Mori fiber space*, or

(b) $dimX = dimY$ then we replace X by Y.

The hope is that by repeating the procedure one ends up either with a variety where K_X is nef (a minimal model) or with a variety of smaller dimension. We have some problems. The first one is that in case (b) above the variety Y may not be smooth, so we must allow varieties to acquire singularities. In higher dimensions, the analogue of Theorem 8 is the Contraction Theorem (see [9, Theorem 3.7]). A second problem is that this theorem is not valid for an arbitrary singular variety, then we must consider varieties with special singularities and specific properties of K_X. The contraction φ_R of a K_X-negative extremal ray R in such varieties can be of three types, refining the classification of φ_R into types (a) and (b) above:

(1) *Mori fiber space* if $dim(Y) < dim(X)$ and the exceptional locus of φ_R (i.e. the points of X where φ_R is not a local isomorphism) is X.

(2) *Divisorial contraction* if φ_R is a birational morphism and its exceptional locus has codimension 1. This is, φ_R contracts an unique divisor prime. In this case $\rho(Y) = \rho(X) - 1$.

(3) *Small contraction* if φ_R is a birational morphism and its exceptional locus has codimension at least 2.

In situation (3), the variety Y has non-allowed singularities and so we cannot replace X by Y. In this case, instead of contracting the K_X-negative extremal ray R, we perform a pseudo-isomorphism $\psi : X \dashrightarrow X^+$ (see definition 19) that substitutes the K_X-negative extremal ray R by a K_{X^+}-positive one R^+ and gives a map $\varphi^+_{R^+} : X^+ \longrightarrow Y$ from φ such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[rr, dashed, "\psi"] \arrow[dr, "\varphi_R"'] & & X^+ \arrow[dl, "\varphi^+_{R^+}"] \\ & Y & \end{tikzcd}$$

Therefore, we replace X by X^+ and continue with the program. Now a third problem appears, we cannot guarantee that the program stops. Whenever we replace the variety we have the three possibilities above. Every time we perform a divisorial contraction the Picard number drops by one. Thus we have only a finite sequence of divisorial contractions. The Picard number remains the same for pseudo-isomorphisms and we cannot guarantee that a sequence of pseudo-isomorphisms ends after finitely many steps. In fact, at present this is an open problem and the most important one of the MMP, known by *Termination of flips*. A good reference for more details is [1] and for a rigorous study of the MMP you can refer to [9].

A simple example of a divisorial contraction is the blow-up of a surface at one point. Now, let us give a formal definition of a pseudo-isomorphism.

Definition 19 (Pseudo-isomophism). A birational map $f : X \dashrightarrow Y$ between projective varieties is called *pseudo-isomorphism* if it is an isomorphism in codimension 1, i.e., there exist open subsets $U \subset X$ and $V \subset Y$ such that $X \backslash U$ and $Y \backslash V$ have codimension $\geqslant 2$ and $f|_U : U \xrightarrow{\sim} V$. In this case we use $f : X \dashrightarrow Y$.

We can easily verify that every pseudo-isomorphism between surfaces is an isomorphism. Moreover, contractions of K_X-negative rays in the MMP for surfaces are either blow-downs or Mori fiber spaces. Thus, in this case we do not have small contractions.

5 The Cremona group in higher dimensions

In this section we discuss the results due to H. Hudson and I. Pan mentioned in the introduction and explore the Sarkisov program for higher dimensions.

5.1 Generators of the Cremona group

As we already mentioned in the introduction, Hilda Hudson's Theorem states that there is no Noether-Castelnuovo Theorem in higher dimensions.

Theorem 9 (Hilda Hudson's Theorem)**.** *For $n \geqslant 3$, $Bir(\mathbb{P}^n)$ cannot be generated by elements of bounded degree.*

A recent and different proof of this theorem was given by Ivan Pan in 1999. In fact he proved a more precise result (Theorem 10). Let us introduce a construction of birational self-maps of $\mathbb{P}^n$ from birational self-maps of $\mathbb{P}^{n-1}$.

Let $P, Q \in \mathbb{C}[x_0, ..., x_n]$ and $R_1, ..., R_n \in \mathbb{C}[x_1, ..., x_n]$ be homogeneous polynomials such that $deg(P) = deg(QR_i)$ for $i = 1, ..., n$. We define the following rational maps:

$$\psi_{P,Q,R} : \mathbb{P}^n \dashrightarrow \mathbb{P}^n, \qquad \psi_{P,Q,R} = (P : QR_1 : ... : QR_n)$$

$$\psi_R : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-1}, \qquad \psi_R = (R_1 : ... : R_n)$$

Assume that $d = deg(P)$, $l = deg(Q)$, $d \geqslant l+1 \geqslant 2$ and P, Q are polynomials such that $P = x_0P_1 + P_2$ and $Q = x_0Q_1 + Q_2$ with $P_1, P_2, Q_1, Q_2 \in \mathbb{C}[x_1, ..., x_n]$ of degree $d-1, d, l-1, l$, respectively and $(P_1, Q_1) \neq (0, 0)$. We have that $\psi_{P,Q,R}$ is birational if and only if so is ψ_R is (see [11, Lemma 2]). Note that the map $\psi_{P,Q,R}$ contracts the hypersurface $Z(Q) \subset \mathbb{P}^n$ to the point $(1 : 0 : ... : 0)$. Conversely, for any hypersurface $Z(Q) \subset \mathbb{P}^n$ of degree l with a point p of multiplicity $\geqslant l - 1$, we can find a birational self-map of $\mathbb{P}^n$ that contracts it to a point (see [11, Corollary 3]).

Theorem 10 (Ivan Pan's Theorem)**.** *For $n \geqslant 3$. Any set of group generators of $Bir(\mathbb{P}^n)$ contains uncountably many elements of unbounded degree.*

We will now give a sketch of the proof.

Proof. First, the set of hypersurfaces that are contracted by a birational self-map of $\mathbb{P}^n$ is finite. Now, consider a family of hypersurfaces X_i of degree $l = 3$ given by the equation $Q_i(x_1, x_2, x_3) = 0$, where every equation $Q_i = 0$ defines a smooth curve C_{Q_i} of degree l on $\{x_0 = x_4 = ... = x_n\}$. Note that X_i is birationally equivalent to $\mathbb{P}^{n-2} \times C_{Q_i}$ and, X_i is birationally equivalent to X_j if and only if C_{Q_i} is isomorphic to C_{Q_j}. Since the set of isomorphism classes of smooth cubics is a 1-parameter family (see [7, Chapter IV, Theorem 4.1] and [7, Chapter IV, Proposition 4.6.1]), we have uncountably many hypersurfaces such that no pair of them are birationally equivalent.

For each i, construct as above a birational map ψ_i which contracts X_i to a point. We can check that if $\varphi_1, ..., \varphi_m$ are birational self-map of $\mathbb{P}^n$ and if $\varphi = \varphi_m \circ ... \circ \varphi_1$ contracts a hypersurface Y, then there exists $i \in \{1, ..., m\}$ and a hypersurface Y_i such that Y_i is birationally equivalent to Y and φ_i contracts Y_i. Therefore, to generate $Bir(\mathbb{P}^n)$ one needs at least as many elements in the set of generators as elements in the family of surfaces $X_i's$.

5.2 Sarkisov program

We will now turn our attention to smooth rational varieties. The outputs of the MMP for smooth rational varieties are Mori fiber spaces.

Definition 20 (Rational variety)**.** A variety X of dimension n is said to be *rational* if it is birationally equivalent to $\mathbb{P}^n$.

Given a smooth rational variety Z, from the MMP we obtain a Mori fiber space X and a sequence of birational morphisms

$$\varphi : Z = X_0 \longrightarrow X_1 \longrightarrow ... \longrightarrow X_n = X \longrightarrow B.$$

However, neither X nor φ are unique. So, it is natural to study birational maps between Mori fiber spaces in the same birational class. The Sarkisov program of section 3.5 can be generalized to any dimension. In this case, the idea is to factor birational maps between Mori fiber spaces into elementary links.

Analogous to the case of surfaces, we have four types of elementary links.

5.2.1 Sarkisov links

Suppose that $X \longrightarrow B$ and $X' \longrightarrow B'$ are two Mori fiber spaces (sometimes we write X/B instead of $X \longrightarrow B$). A Sarkisov link $\chi : X \dashrightarrow X'$ is of one of four types:

$$
\begin{tikzcd}
Z \arrow[rr, dotted] \arrow[d] & & X' \arrow[d] \\
X \arrow[urr, dashed] \arrow[dr] & & B' \arrow[dl] \\
& B &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
Z \arrow[rr, dotted] \arrow[d] & & Z' \arrow[d] \\
X \arrow[rr, dashed] \arrow[dr] & & X' \arrow[dl] \\
& B = B' &
\end{tikzcd}
$$

I *II*

$$
\begin{tikzcd}
X \arrow[rr, dotted] \arrow[d] \arrow[drr, dashed] & & Z \arrow[d] \\
B \arrow[dr] & & X' \arrow[dl] \\
& B' &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X \arrow[rr, dashed] \arrow[d] & & X' \arrow[d] \\
B \arrow[dr] & & B' \arrow[dl] \\
& C &
\end{tikzcd}
$$

III *IV*

where the pseudo-isomorphism $\dashrightarrow$ is a composition of flips, flops or anti-flips (see [3]); the maps $Z \longrightarrow X$ and $Z' \longrightarrow X'$ are divisorial contractions; and the maps $B \longrightarrow B'$, $B' \longrightarrow B$, $B \longrightarrow C$ and $B' \longrightarrow C$ are birational morphisms with relative Picard number 1 (see [3]).

Note that the corresponding diagram of Sarkisov links for surfaces are:

$$
\begin{tikzcd}
\mathbb{F}_1 \arrow[rr, equal] \arrow[d] & & \mathbb{F}_1 \arrow[d] \\
\mathbb{P}^2 \arrow[urr, dashed] \arrow[dr] & & \mathbb{P}^1 \arrow[dl] \\
& pt &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
Z \arrow[rr, equal] \arrow[d] & & Z \arrow[d] \\
\mathbb{F}_m \arrow[rr, dashed] \arrow[dr] & & \mathbb{F}_{m\pm1} \arrow[dl] \\
& \mathbb{P}^1 &
\end{tikzcd}
$$

I *II*

$$
\begin{tikzcd}
\mathbb{F}_1 \arrow[r, equal] \arrow[d] \arrow[dr] & \mathbb{F}_1 \arrow[d] \\
\mathbb{P}^1 \arrow[dr] & \mathbb{P}^2 \arrow[d] \\
& pt
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{P}^1 \times \mathbb{P}^1 \arrow[rr] \arrow[d] & & \mathbb{P}^1 \times \mathbb{P}^1 \arrow[d] \\
\mathbb{P}^1 \arrow[dr] & & \mathbb{P}^1 \arrow[dl] \\
& pt &
\end{tikzcd}
$$

III $\qquad\qquad$ IV

Theorem 11 (Sarkisov program)**.** *Every birational map $\varphi : X \dashrightarrow X'$ between Mori fiber spaces $X \longrightarrow B$ and $X' \longrightarrow B'$ is a composition of Sarkisov links.*

$$
\begin{tikzcd}
X = X_0 \arrow[r, dashed, "\psi_1"'] \arrow[d] \arrow[rrr, dashed, bend left, "\varphi"] & X_1 \arrow[r, dashed, "\psi_2"'] \arrow[d] & \cdots \arrow[r, dashed, "\psi_k"'] & X_k = X' \arrow[d] \\
B = B_0 & B_1 & & B_k = B'
\end{tikzcd}
$$

The algorithm for this program is the same as described in Theorem 7. We start by assigning φ a Sarkisov degree $s.deg(\varphi)$ and searching a link ψ_1 such that $s.deg(\varphi \circ \psi_1^{-1}) < s.deg(\varphi)$. We continue inductively with ψ_1 in place of φ.

Consider a birational map $\varphi : X \dashrightarrow X'$ between Mori fiber spaces $X \longrightarrow B$ and $X' \longrightarrow B'$. Its decomposition into Sarkisov links is not unique in general and two such decompositions define a relation (see definition 21) in the Sarkisov program. If

$$\varphi : X/B = X_0/B_0 \dashrightarrow X_1/B_1 \dashrightarrow \cdots \dashrightarrow X_k/B_k = X'/B'$$

and

$$\varphi : X/B = \tilde{X}_0/\tilde{B}_0 \dashrightarrow \tilde{X}_1/\tilde{B}_1 \dashrightarrow \cdots \dashrightarrow \tilde{X}_l/\tilde{B}_l = X'/B'$$

are two different decomposition of φ, then

$$X/B = X_0/B_0 \dashrightarrow \cdots \dashrightarrow X_k/B_k \cong \tilde{X}_l/\tilde{B}_l \dashrightarrow \cdots \dashrightarrow \tilde{X}_0/\tilde{B}_0 = X/B$$

is an element of $Aut(X)$ which commutes with the map $X \longrightarrow B$.

Definition 21 (Relations)**.** A non-trivial *relation* in the Sarkisov program is a composition of $k > 2$ Sarkisov links

$$\psi_k \circ \cdots \circ \psi_1 \in Aut(X_1) \tag{2}$$

which define an automorphism of $X_{k+1}/B_{k+1} = X_1/B_1$ that commutes with the map $X_1 \longrightarrow B_1$, where $\psi_i : X_i/B_i \dashrightarrow X_{i+1}/B_{i+1}$.

Definition 22 (Elementary relations)**.** The relation (2) is *elementary* if no proper subchains of links forms a relation and $\psi_{i+1} \circ \psi_i$ is not a Sarkisov link, for all $i = 1, ..., k-1$.

6 Proof strategy for Theorem 1

Finally, we will study Theorem 1 focusing on the techniques and tools which were used in its proof. As we have seen in Theorem 5, the Jonquière maps generate the Cremona group $Bir(\mathbb{P}^2)$. Theorem 1 says that this statement does not hold in higher dimensions.

We are interested in constructing a surjective group homomorphism from $Bir(\mathbb{P}^n)$ to $\mathbb{Z}/2\mathbb{Z}$. The natural way for constructing a group homomorphism $f : G \longrightarrow H$ is to take a set of generators of G and define its image by f in H, so that the relations between such generators are preserved. So far, a specific set of generators of $Bir(\mathbb{P}^n)$ is not known for $n \geqslant 3$. By Theorem 11, we know that every birational self-map of $\mathbb{P}^n$ is a composition of Sarkisov links. However, these links do not generate $Bir(\mathbb{P}^n)$, in fact, the Sarkisov links do not belong to the Cremona group. This motivates us to consider the set $BirMori(\mathbb{P}^n)$ of birational maps between Mori fiber spaces birational to $\mathbb{P}^n$. Note that two elements $\varphi, \varphi' \in BirMori(\mathbb{P}^n)$, $\varphi : X \dashrightarrow Y$ and $\varphi' : X' \dashrightarrow Y'$, can only be composed if $Y = X'$. Therefore, $BirMori(\mathbb{P}^n)$ is not a group but has a groupoid structure.

Definition 23 (Groupoid)**.** Let $\mathcal{G}$ be a set endowed with a product map

$$\mathcal{G}^2 \longrightarrow \mathcal{G}, \qquad\qquad (g, h) \longmapsto gh,$$

where the set $\mathcal{G}^2 \subset \mathcal{G} \times \mathcal{G}$ is called the set of composable pairs, and an inverse map

$$\mathcal{G} \longrightarrow \mathcal{G}, \qquad\qquad g^{-1},$$

such that for all $f, g, h \in \mathcal{G}$ the following conditions hold:

i. $(g^{-1})^{-1} = g$;

ii. If $(g, h), (h, l) \in \mathcal{G}^2$, then $(gh, l), (g, hl) \in \mathcal{G}^2$ and $(gh)l = g(hl)$;

iii. $(g^{-1}, g) \in \mathcal{G}^2$ and if $(g, h) \in \mathcal{G}^2$, then $g^{-1}(gh) = h$;

iv. $(g, g^{-1}) \in \mathcal{G}^2$ and if $(l, g) \in \mathcal{G}^2$, then $(lg)g^{-1} = l$.

Example 16. Let $X \longrightarrow B$ be a Mori fiber space. The set of birational maps between Mori fiber spaces birational to X, denoted by $BirMori(X)$ is a groupoid. Note that $Bir(X)$ is a subgroupoid of $BirMori(X)$.

The next result gives a representation of groupoid $BirMori(X)$.

Theorem 12. *Let $X \longrightarrow B$ be a Mori fiber space.*

i) The groupoid $BirMori(X)$ is generated by Sarkisov links and automorphisms.

ii) Any relation between Sarkisov links in $BirMori(X)$ is generated by elementary relations.

For a proof you can refer to [3, Theorem 4.28]. In particular, this theorem holds for $X = \mathbb{P}^n$.

The strategy of the proof. Let $n \geqslant 3$. Let us describe how we can obtain the group homomorphism $Bir(\mathbb{P}^n) \longrightarrow \mathbb{Z}/2\mathbb{Z}$ from Theorem 1. From Sarkisov program and Theorem 12 we have that any birational map between two Mori fiber spaces is a composition of Sarkisov links and the relations between Sarkisov links are generated by elementary relations. So, we proceed to construct the groupoid $BirMori(\mathbb{P}^n)$ where Sarkisov links and elementary relations give it a representation. The group $Bir(\mathbb{P}^n)$ is a subgroupoid of $BirMori(\mathbb{P}^n)$. Now, we construct a groupoid homomorphism from $BirMori(\mathbb{P}^n)$ to a free product of $\mathbb{Z}/2\mathbb{Z}$ whose restriction to $Bir(\mathbb{P}^n)$ gives the desired group homomorphism. The automorphisms and every Sarkisov link are mapped to identity (except some special links) by the groupoid homomorphism and the elementary relations are preserved. Restricting this groupoid homomorphism to $Bir(\mathbb{P}^n)$ and by Theorem E in [3], we have a surjective group homomorphism

$$Bir(\mathbb{P}^n) \xrightarrow{\tau} \mathop{\ast}_{J} \mathbb{Z}/2\mathbb{Z},$$

where the indexing set J has the same cardinality as $\mathbb{C}$. By construction, automorphisms of $\mathbb{P}^n$ are sent onto the identity. It is possible to construct τ so that the Jonquière maps are in its kernel. For that, a qualitative description of the decomposition of the Jonquière maps by elementary links is used. So,

$$|\tau(G)| \leqslant |S| < |J|,$$

therefore there exists some factor $\mathbb{Z}/2\mathbb{Z}$ such that it is not in $\tau(G)$. Thus, we make the composition of τ with the projection onto this factor and therefore we achieve the desired homomorphism.

You can see more details about the tools used and the complete proof in [3]. As consequence of Theorem 1, the group $Bir(\mathbb{P}^n)$ is not simple, this result was not known until then for $n \geqslant 3$.

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